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Concepts—FEM Package for Elliptic Problems in 2D and 3D

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Concept Oriented Design

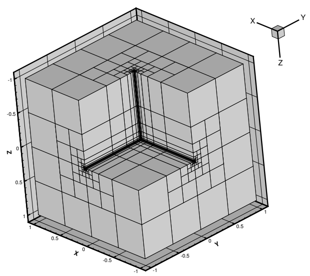
The numerical methods which should be implemented are already formulated in an abstract way based on hierarchical structured mathematical concepts. Therefore, represent each concept by a module and combine the modules according to the numerical algorithm. This defines concept oriented design.

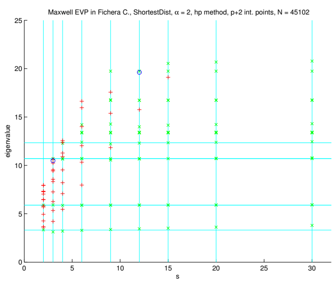
The Software Package CONCEPTS

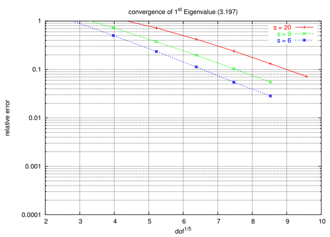
CONCEPTS [3] is designed using *concept oriented design* and written nearly completely in C++. It is used for BEM [4], *hp* FEM and DGFEM.

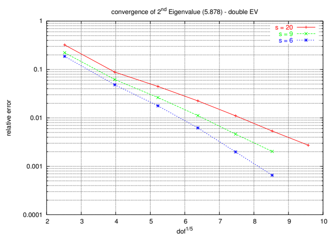
Results: Exponential Convergence

Solving the Maxwell Eigen value problem in a perfect conductor boundary condition domain $\Omega = (-1,1)^3 \setminus (-1,0)^3$ (so called Fichera corner) with $\varepsilon = \mu = 1$ using weighted regularization [1].









Mathematical Concepts

Find $u \in H^1_{\Gamma_D}(\Omega)$ such that $\forall v \in H^1_{\Gamma_D}(\Omega)$:

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega uv \, dx}_{=:a(u,v) \text{ bilinear form}} = \underbrace{\int_\Omega f v \, dx + \int_{\Gamma_N} g_N v \, ds}_{=:l(v) \text{ linear form}} .$$

Find $u_N \in V_N := S^{p,1}_{\Gamma_D}(\Omega, \mathcal{T})$ such that $\forall v \in V_N$:

$$a(u,v) = l(v).$$

Using a basis $\{\Phi_i\}_{i=1}^N$ of V_N makes it possible to obtain a stiffness matrix $[\underline{A}]_{ij} = a(\Phi_i, \Phi_j)$ and a load vector $[\boldsymbol{l}]_i = l(\Phi_i)$. The resulting linear system $\underline{A}\boldsymbol{u}_N = \boldsymbol{l}$ can be solved with a linear solver (*e. g.* conjugate gradients).

Assembling using T matrices: $\Phi_i|_K = \sum_{j=1}^{m_K} [\underline{T}_K]_{ji} \varphi_j^K$

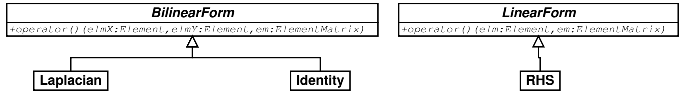
$$\boldsymbol{l} = l(\boldsymbol{\Phi}) = \sum_{\tilde{K}} \underline{T}_{\tilde{K}}^\top l(\boldsymbol{\varphi}^{\tilde{K}}) = \sum_{\tilde{K}} \underline{T}_{\tilde{K}}^\top \boldsymbol{l}_{\tilde{K}}$$

$$\underline{A} = a(\boldsymbol{\Phi}, \boldsymbol{\Phi}) = \sum_{K,\tilde{K}} \underline{T}_{\tilde{K}}^\top a(\boldsymbol{\varphi}^K, \boldsymbol{\varphi}^{\tilde{K}}) \underline{T}_K = \sum_{K,\tilde{K}} \underline{T}_{\tilde{K}}^\top \underline{A}_{\tilde{K}K} \underline{T}_K$$

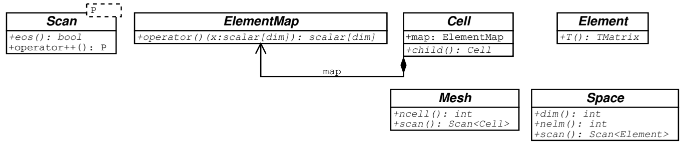
The constraints of hanging nodes are eliminated using S matrices: $\varphi_j^K \Big|_{K'} = \sum_{l=1}^{m_{K'}} [\underline{S}_{K'K}]_{lj} \varphi_l^{K'}$ for a $K' \subset K$ generated by a subdivision. $\underline{T}_{K'} = \underline{S}_{K'K} \underline{\tilde{T}}_K + \underline{\tilde{T}}_{K'}$ builds the T matrix for K'. The S matrices for quads and hexahedra can be built with tensor products from 1D S matrices. This makes anisotropic refinements possible.

Fundamental Classes

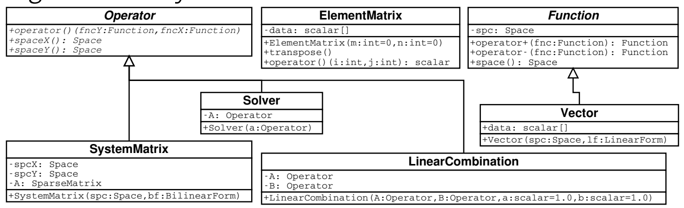
Bilinear form $a(.,.)$ and linear form $l(.)$:



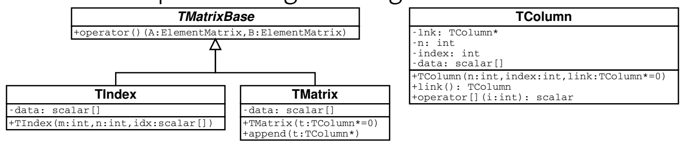
Mesh $\mathcal{T}$ and space V_N:



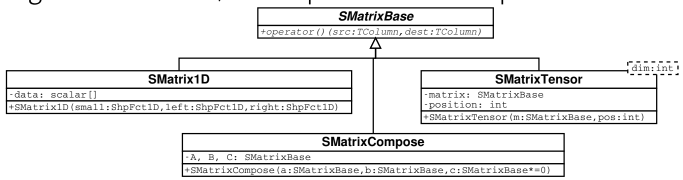
Assembling the global matrix and the load vector and solving the linear system:



The T matrices are built columnwise: each column in a T matrix corresponds to a global degree of freedom:



S matrices in 1D are computed by solving a linear system. In higher dimensions, tensor product and composition is used:



References

[1] M. COSTABEL, M. DAUGE. Weighted regularization of Maxwell equations in polyhedral domains. *Numer. Math.* **93**(2) (2002) 239–277.
[2] P. FRAUENFELDER, C. LAGE. Concepts—an object-oriented software package for partial differential equations. *M2AN* **36**(5) (2002) 937–951.
[3] C. LAGE, P. FRAUENFELDER, G. SCHMIDLIN ET. AL. *Concepts—C++ Class Library for Elliptic Problems*, http://www.math.ethz.ch/~concepts/.
[4] G. SCHMIDLIN. *Fast Solution Algorithms for Integral Equations in* $\mathbb{R}^3$. PhD thesis, Swiss Federal Institute of Technology Zurich, 2003.