Computing Maxwell Eigenvalues in 3D using H^1 conforming FEM

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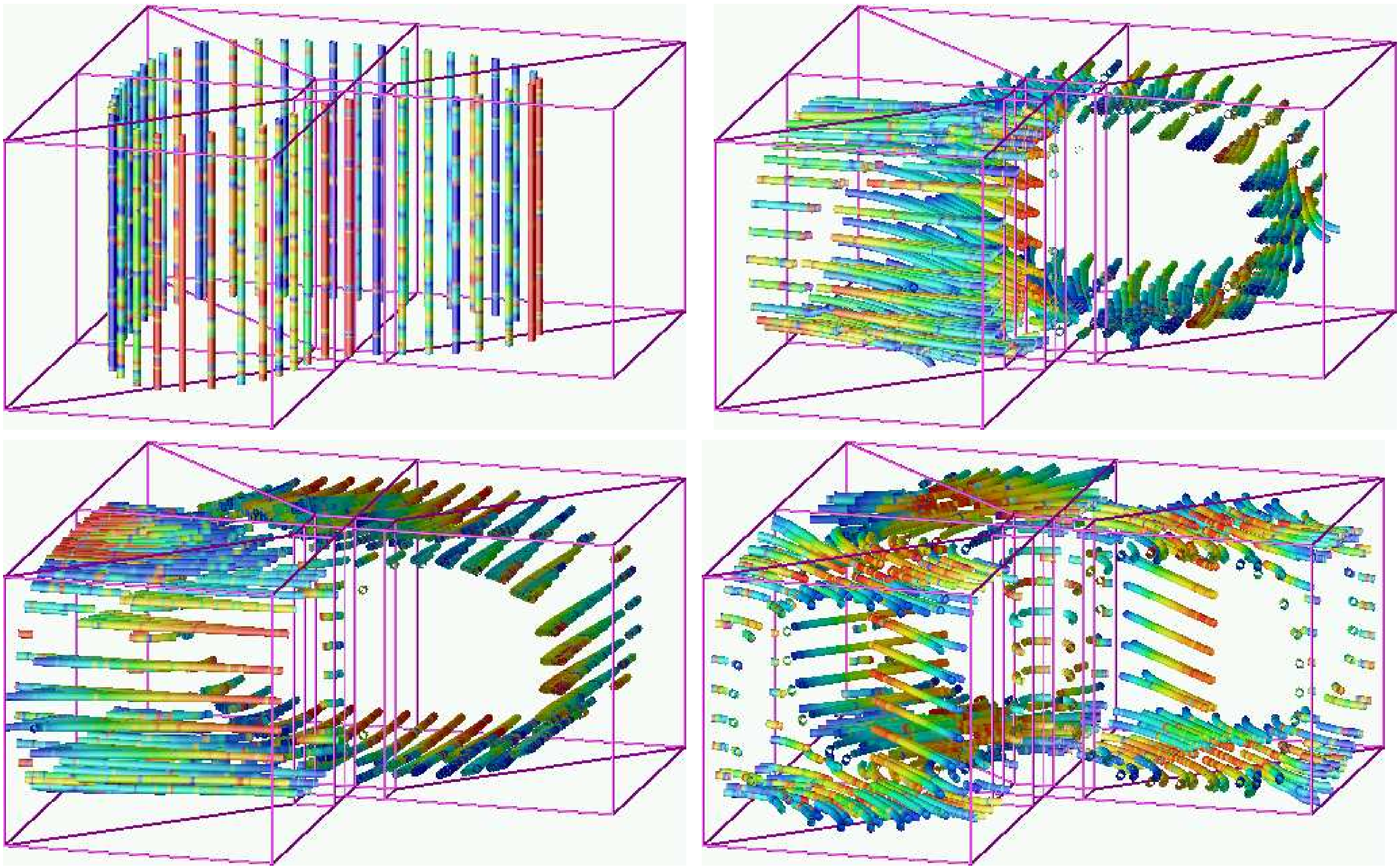
Motivation



Overview

- Introduction: FEM & Exponential Convergence
- Assembling
- Handling Hanging Nodes
- Finding Regular Supports
- Maxwell Eigenvalue Problems
- Perspectives

Our Software: Concepts

- Started by Christian Lage during his Ph.D. studies (1995).
- Used and improved by Frauenfelder, Matache, Schmidlin, Schmidt and several students.
- Concept Oriented Design using mathematical principles [1].
- Currently two parts: hp-FEM, BEM (wavelet and multipole methods).
- C++ class library for general elliptic PDEs.

[1] P. F. and Ch. Lage, "Concepts—An Object Oriented Software Package for Partial Differential Equations", *Mathematical Modelling and Numerical Analysis* 36 (5), pp. 937–951 (2002).

FEM Basics

Elliptic Boundary Value Problem in $\Omega \subset \mathbb{R}^n$ in variational form:
Find $u \in V$ such that:

$$a(u, v) = l(v) \qquad \forall v \in V,$$

where V is a FE space $a(.\,,.)$ a bilinear form and $l(.)$ a linear form.

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Example: $-\Delta u + u = f$ in Ω and $u = 0$ on $\partial\Omega$

$$\Longrightarrow a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega uv \, dx,$$

$$l(v) = \int_\Omega fv \, dx$$

$$V = S^{1,\underline{p}}(\Omega, \mathcal{T}) \subset H^1(\Omega).$$

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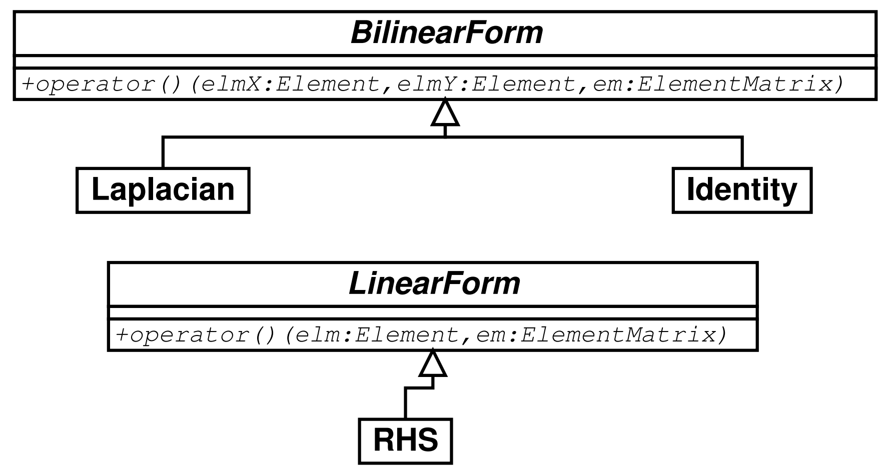
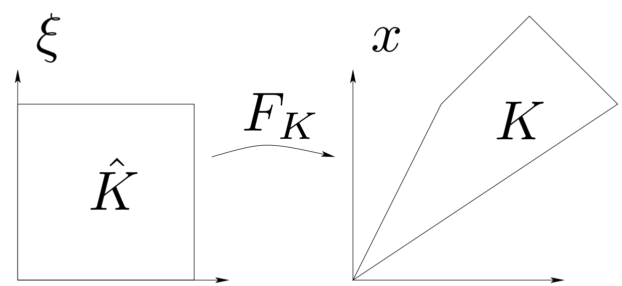
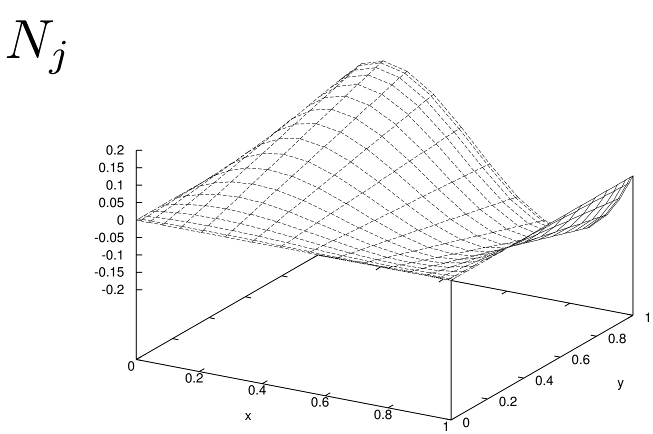


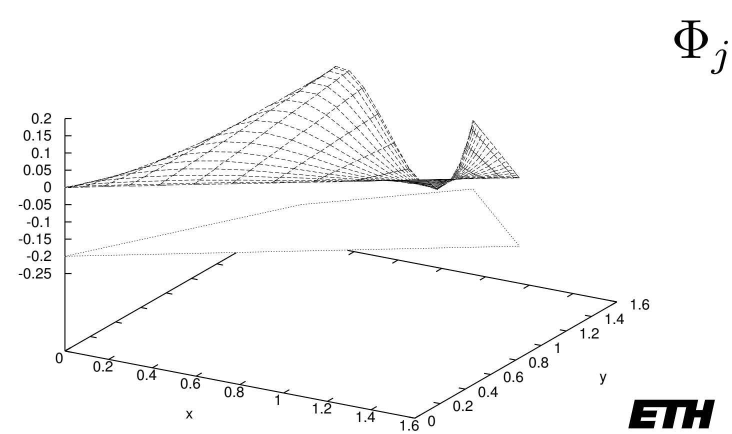
FE Space

- Basis $\{\Phi_i\}_{i=1}^N$ constructed from element shape functions ϕ_j^K on elements $K \in \mathcal{T}$.
- Reference element shape functions: N_j, element map: $F_K : \hat{K} \to K$

$$\Rightarrow \phi_j^K \circ F_K = N_j .$$

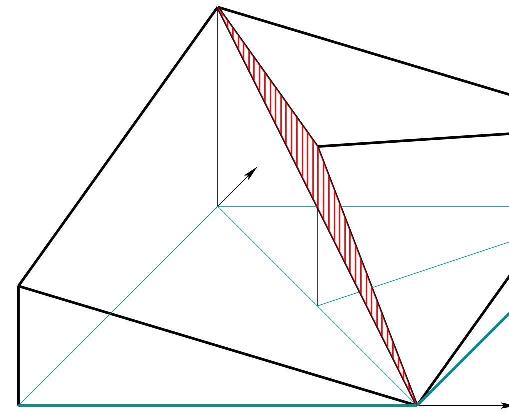


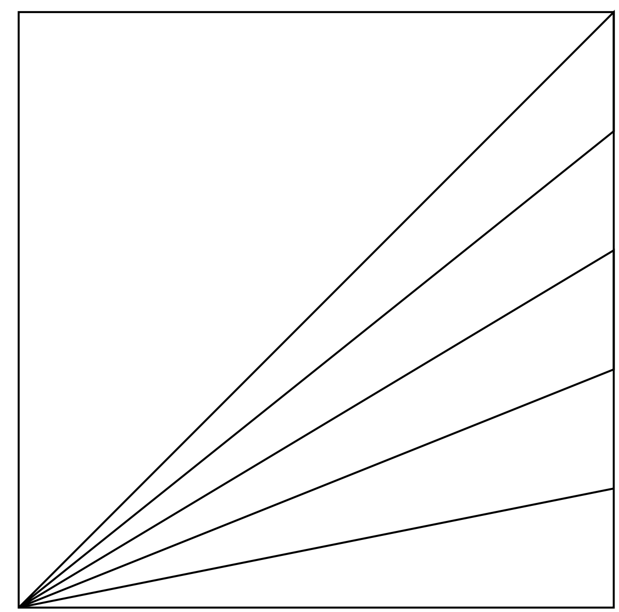




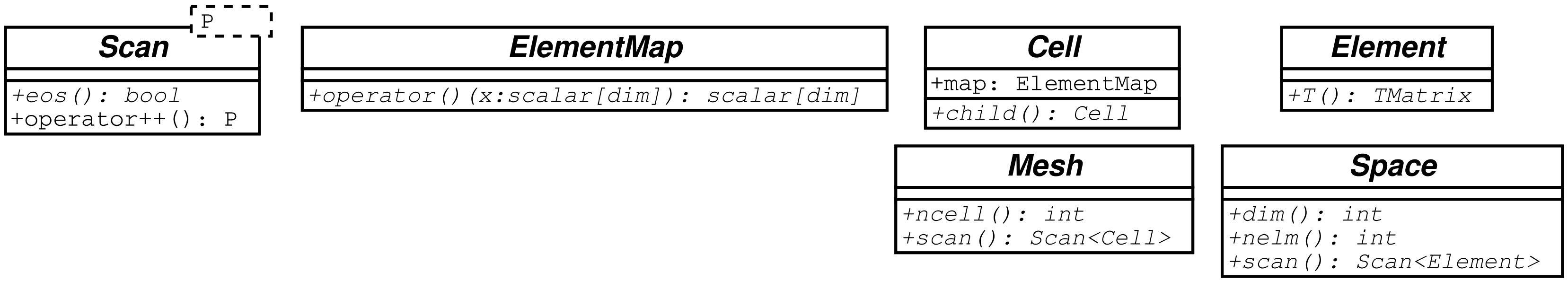
FE Meshes

- Need to be regular (no hanging nodes):
- Need to be shape-regular (no degenerate elements):





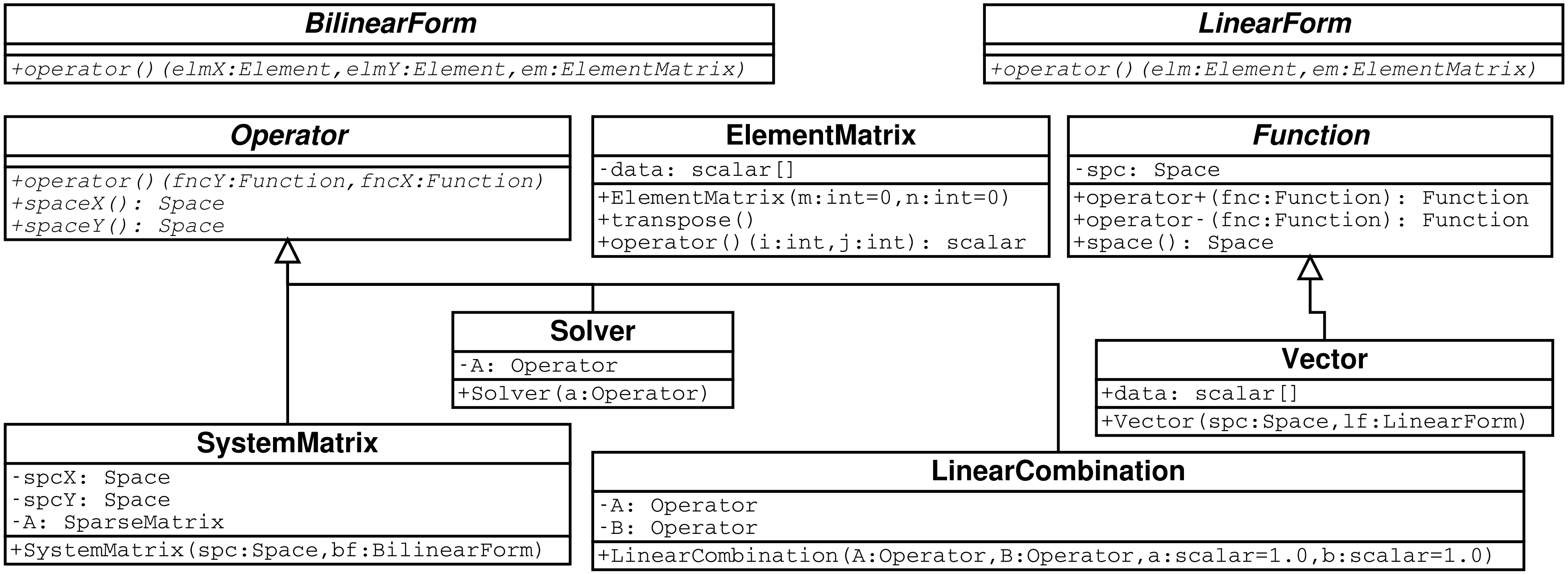
Classes for Mesh & Space



Ideas:

- Mesh consists of Cells, every Cell has an ElementMap
- Space consists of Elements, every Element has a Cell
- Scan used to loop over Elements of a Space or Cells of a Mesh

Classes for PDEs



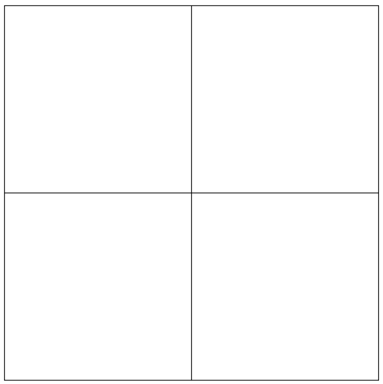
Variational formulation of an elliptic PDE on Ω:
Find $u \in V$ such that

$$a(u,v) = l(v) \qquad \forall v \in V$$

$$\Rightarrow (\boldsymbol{A} + \boldsymbol{M})\underline{u}_N = \underline{l}_N$$ solve for $\underline{u}_N$.

h *Refinements*

In general: elements with large error should be modified somehow
$\leadsto$ h refinement



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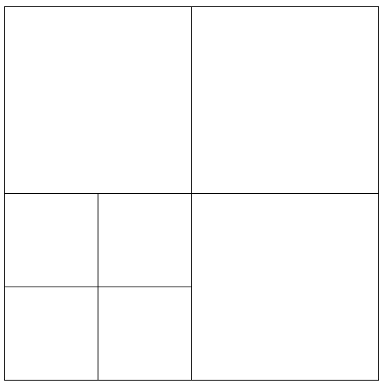

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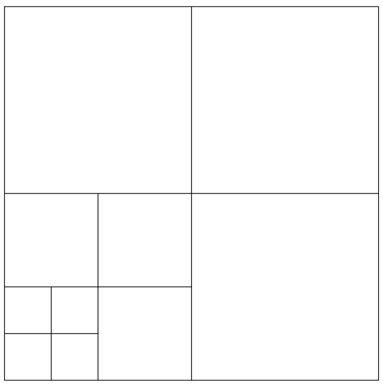

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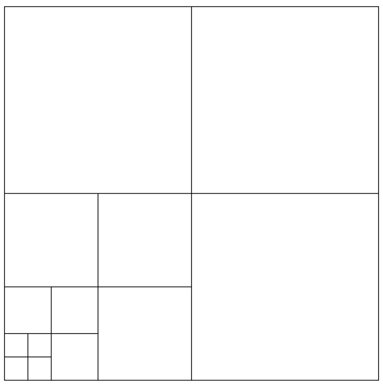

1-irregular meshes!

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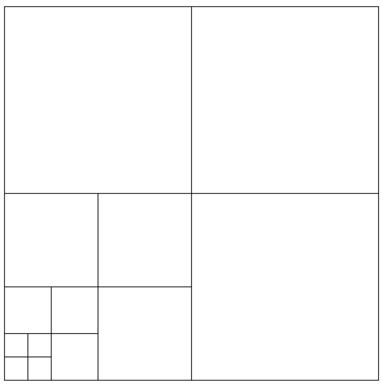

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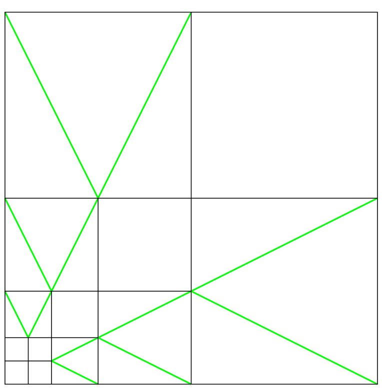
Remedy:



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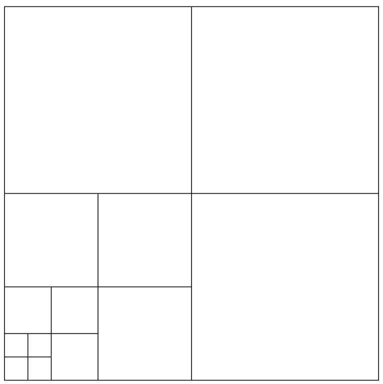

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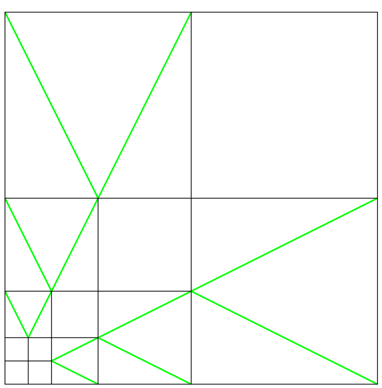


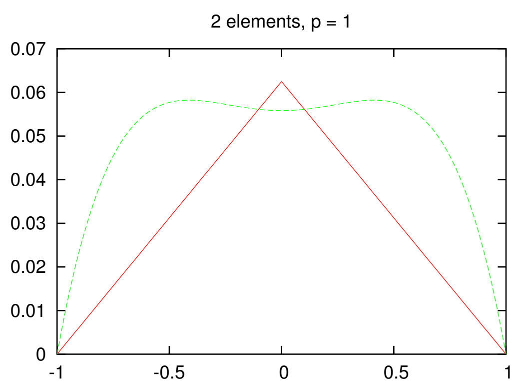
or

Handling the hanging nodes by applying constraints to them: A hanging node is not a real degree of freedom.

p *Enrichment*

Instead of refining an element: increasing polynomial degree p. Good for smooth functions.

h FEM:



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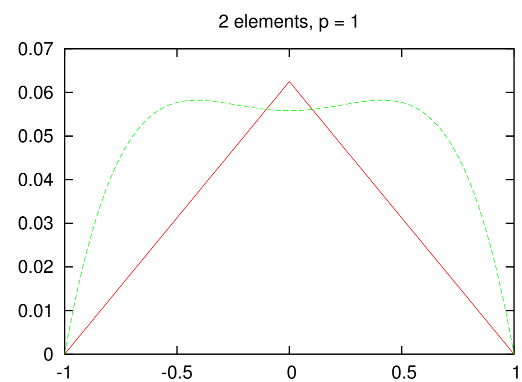
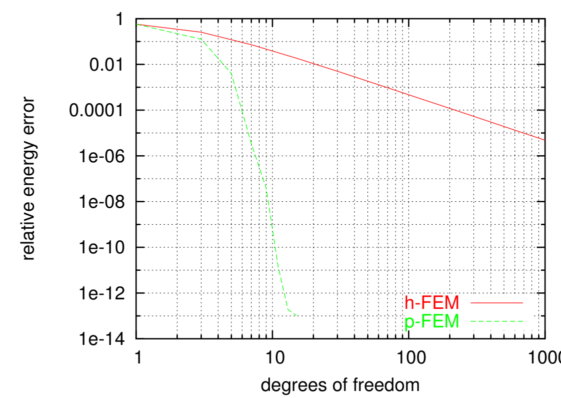


Convergence:



Solving the problem

$$-\Delta u + u = x^2 \text{ in } \Omega = (-1,1)$$

$$u = 0 \text{ on } \partial\Omega = \{-1,1\},$$

$$\Rightarrow u(x) = -3\frac{\cosh(x)}{\cosh(1)} + x^2 + 2.$$

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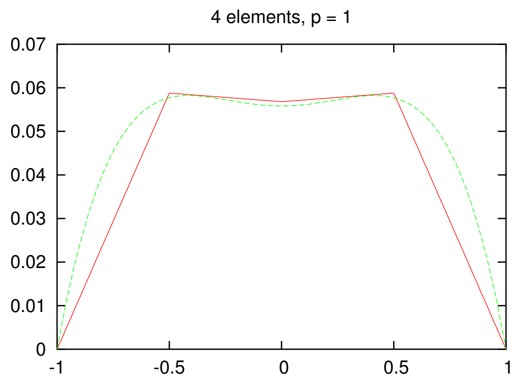

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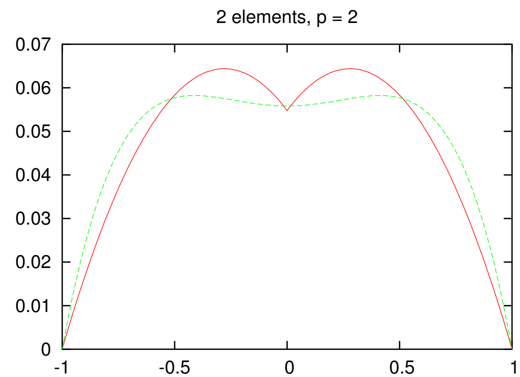

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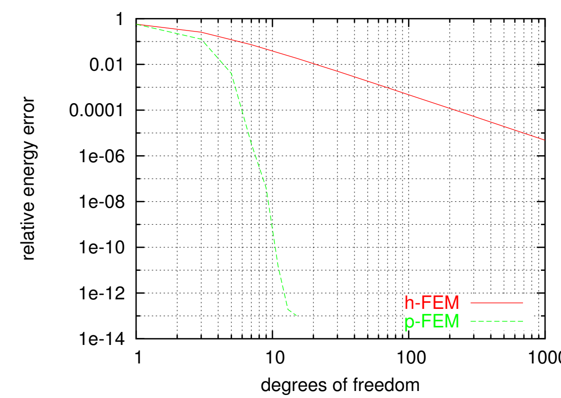

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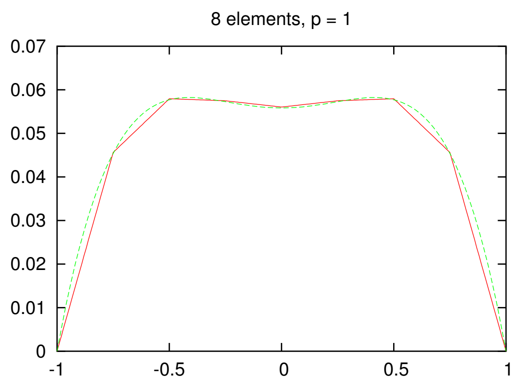

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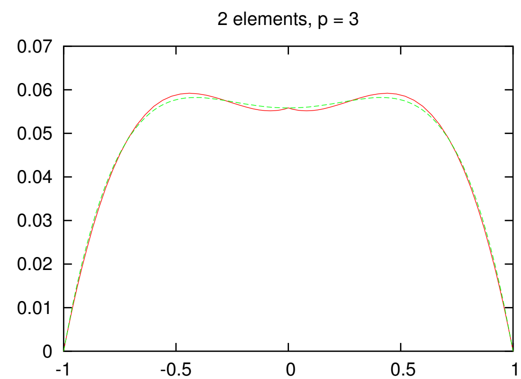

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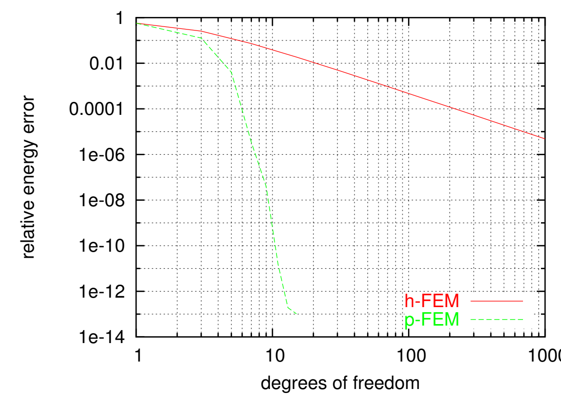

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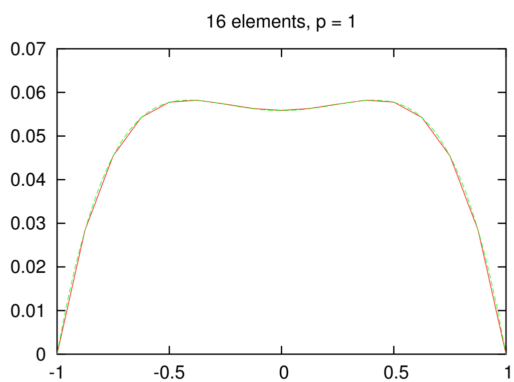

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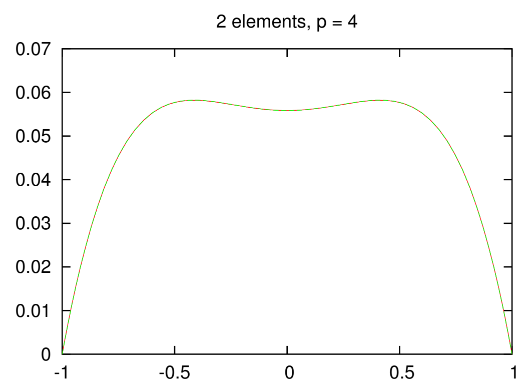

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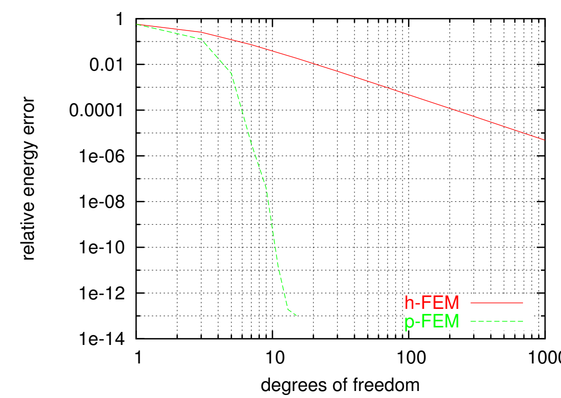

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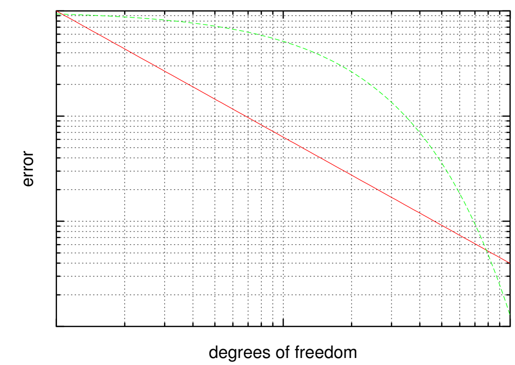
Exponential Convergence

Theorem: Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ a polyhedron, $V_N = S^{1,\underline{p}}(\Omega, \mathcal{T}) \ni u_N$ the FE space and $u \in H^k(\Omega)$, $k \geq 1$ the exact solution.

Then: $\|u - u_N\| \leq c \left(\frac{h}{p}\right)^{\min(p,k-1)} \|u\|_{H^k(\Omega)}$.



h FEM: $\|u - u_N\| \leq c_1 h^{c_2}$ algebraic convergence

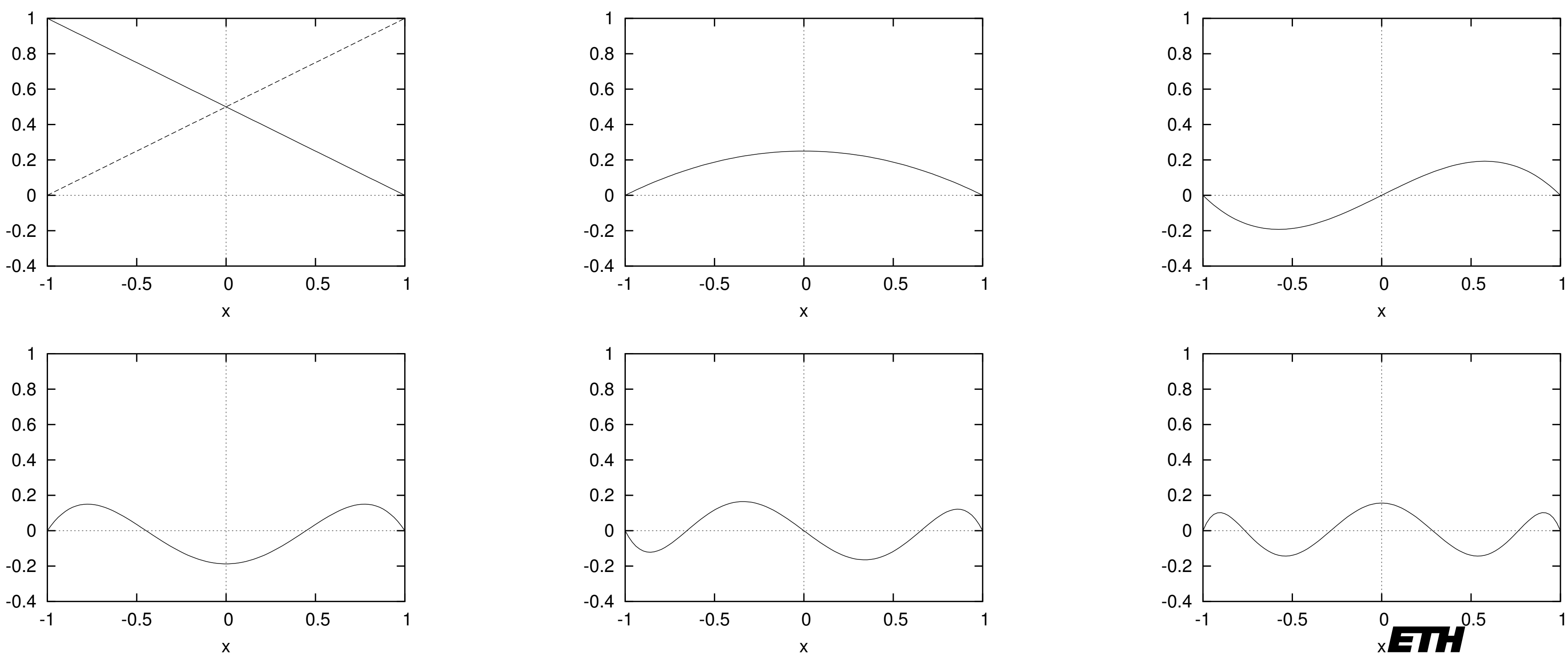
p FEM: $\|u - u_N\| \leq c \left(\frac{1}{p}\right)^p$ if $u \in \mathcal{C}^\infty$ exponential convergence

hp FEM: close to singularities: h FEM
in regular areas: p FEM
$\Rightarrow \|u - u_N\| \leq c \exp(-bN^\alpha)$ if $u \in B^2_\beta$ exponential convergence
$\alpha = 3$ in $\mathbb{R}^2$
$\alpha = 5$ in $\mathbb{R}^3$

Shape Functions in 1D

The reference element shape functions on $(-1, 1)$ of order p:

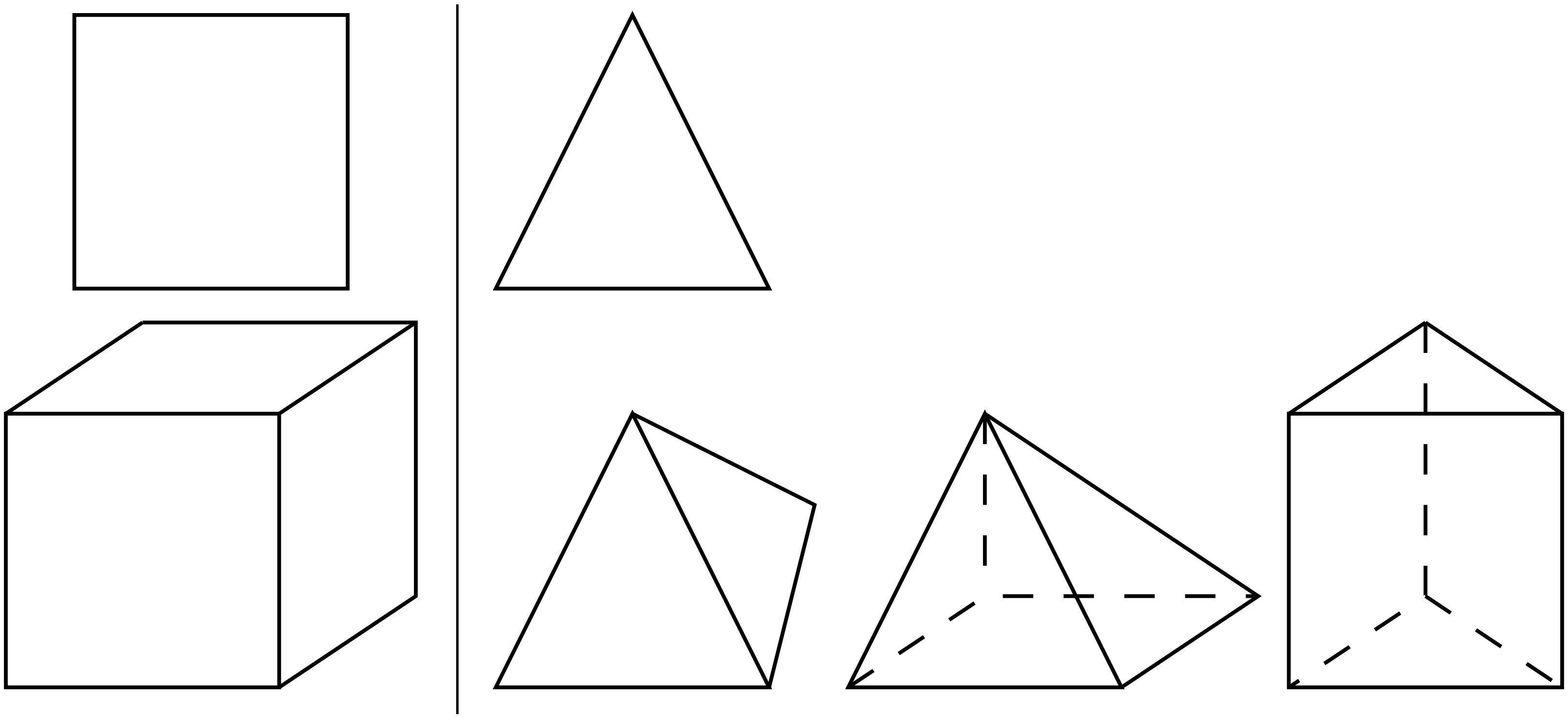
$$N_i(\xi) = \begin{cases} \frac{1-\xi}{2} & i = 0 \\ \frac{1-\xi}{2}\frac{1+\xi}{2} P_{i-1}^{1,1}(\xi) & 1 \le i \le p-1 \\ \frac{1+\xi}{2} & i = p \end{cases}$$



Element Types in 2D and 3D

2D:

3D:



Quads and Hexahedra: fully anisotropic, variable degree p.

Shape Functions in 2D

The reference element shape functions on $(-1,1)^2$ of order (p,q) are tensor product functions of 1D shape functions:

$$N_{i,j} = N_i \otimes N_j$$
$$N_{i,j}(\xi,\eta) = N_i(\xi)\cdot N_j(\eta)$$

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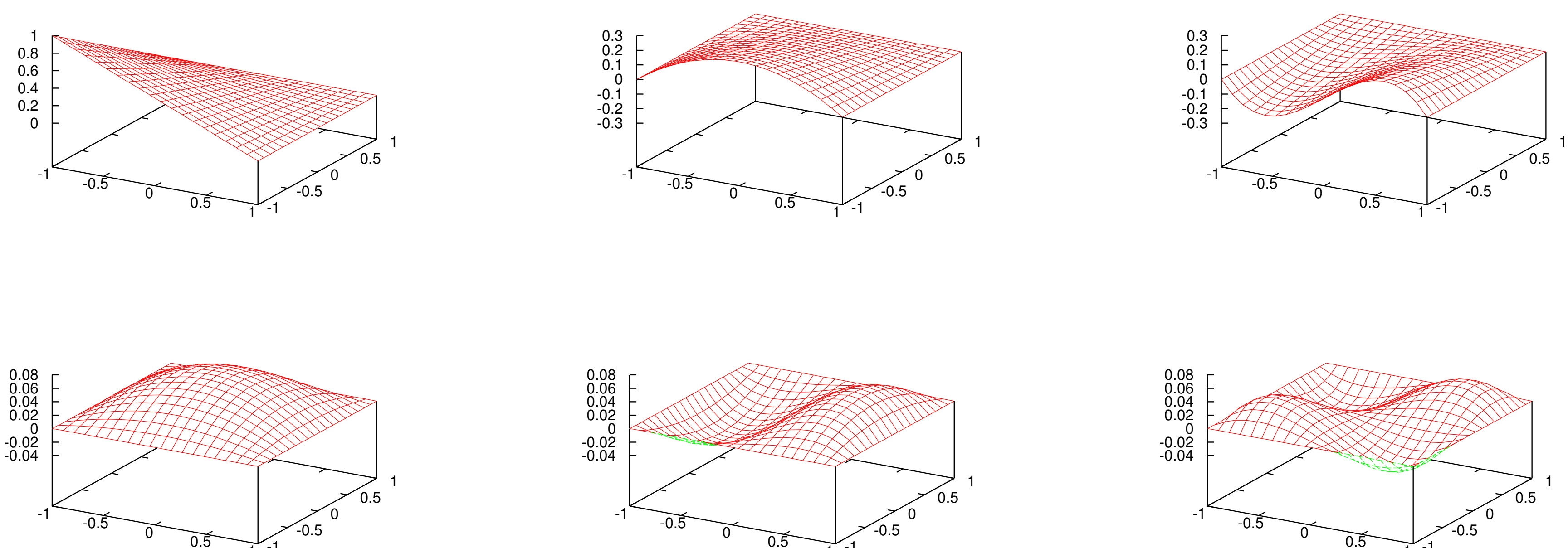

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T Matrix

Definition 1 (T Matrix). *Element shape functions* $\{\phi_j^K\}_{j=1}^{m_K}$ *on element* K*, global basis functions* $\{\Phi_i\}_{i=1}^{N}$*.*
The T matrix $\boldsymbol{T}_K \in \mathbb{R}^{m_K \times N}$ *of element* K *is implicitly defined by*

$$\Phi_i|_K = \sum_{j=1}^{m_K} [\boldsymbol{T}_K]_{ji}\, \phi_j^K$$

as vectors:

$$\underline{\Phi}|_K = \boldsymbol{T}_K^\top \underline{\phi}^K .$$

Assembly using T Matrices

Stiffness matrix: $A_{ij} = a(\phi_i, \phi_j)$, load vector: $l_i = l(\phi_i)$.
Assembling:

$$\underline{l} = l(\underline{\Phi}) = l\Big(\sum_{\tilde{K}} \boldsymbol{T}_{\tilde{K}}^{\top} \underline{\phi}^{\tilde{K}}\Big) = \sum_{\tilde{K}} \boldsymbol{T}_{\tilde{K}}^{\top} l(\underline{\phi}^{\tilde{K}}) = \sum_{\tilde{K}} \boldsymbol{T}_{\tilde{K}}^{\top} \underline{l}_{\tilde{K}}$$

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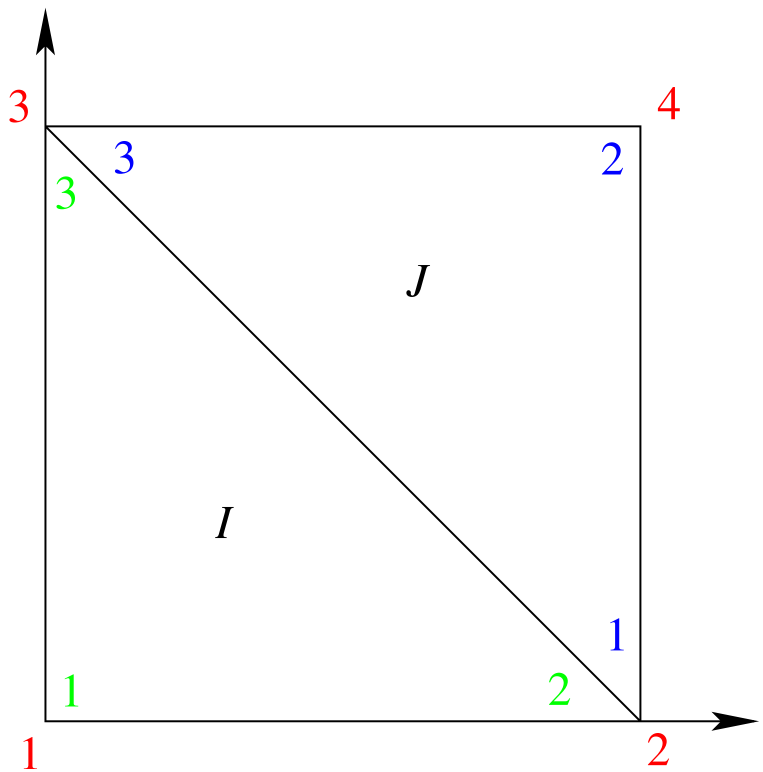
$$\boldsymbol{A} = a(\underline{\Phi}, \underline{\Phi}) = \sum_{K,\tilde{K}} \boldsymbol{T}_{\tilde{K}}^{\top} a(\underline{\phi}^{K}, \underline{\phi}^{\tilde{K}}) \boldsymbol{T}_K = \sum_{K,\tilde{K}} \boldsymbol{T}_{\tilde{K}}^{\top} \boldsymbol{A}_{\tilde{K}K} \boldsymbol{T}_K$$

Note: $\boldsymbol{A}_{\tilde{K}K} = 0$ in standard FEM for $\tilde{K} \neq K$.

Example 1: Regular Mesh

Two elements with three local shape functions each and four global basis functions.



$$\boldsymbol{T}_I = \begin{pmatrix} & 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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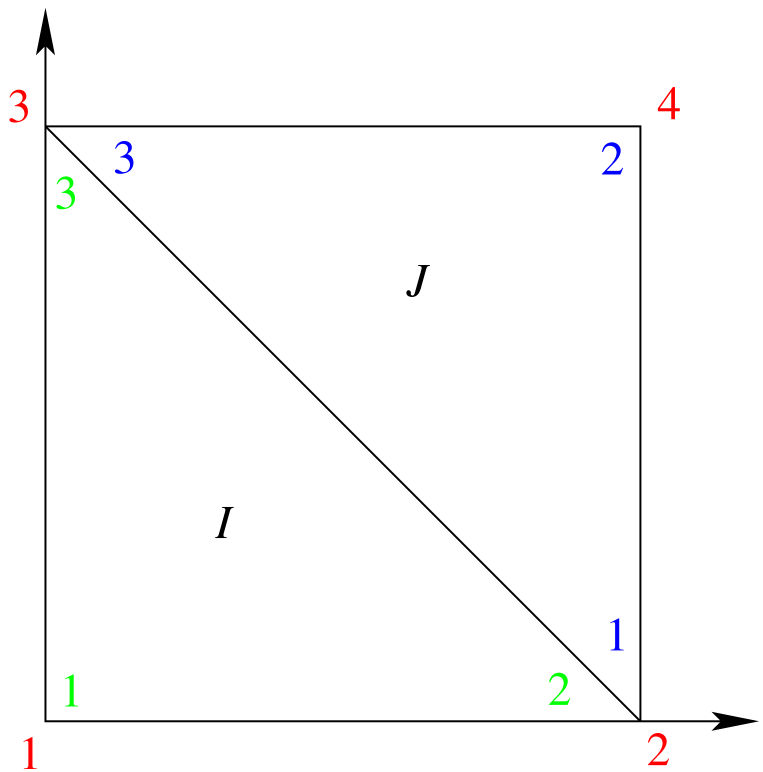

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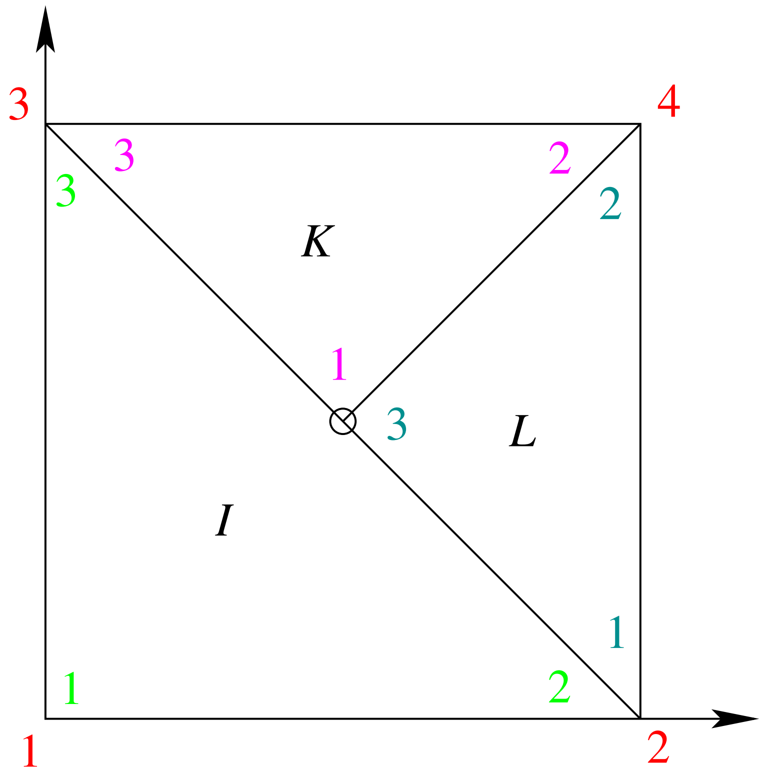
$$\boldsymbol{T}_J = \begin{pmatrix} & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Example 2: Irregular Mesh

Three elements with three local shape functions each and four global basis functions. The hanging node is marked with $\circ$.



$$\boldsymbol{T}_L = \begin{pmatrix} & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

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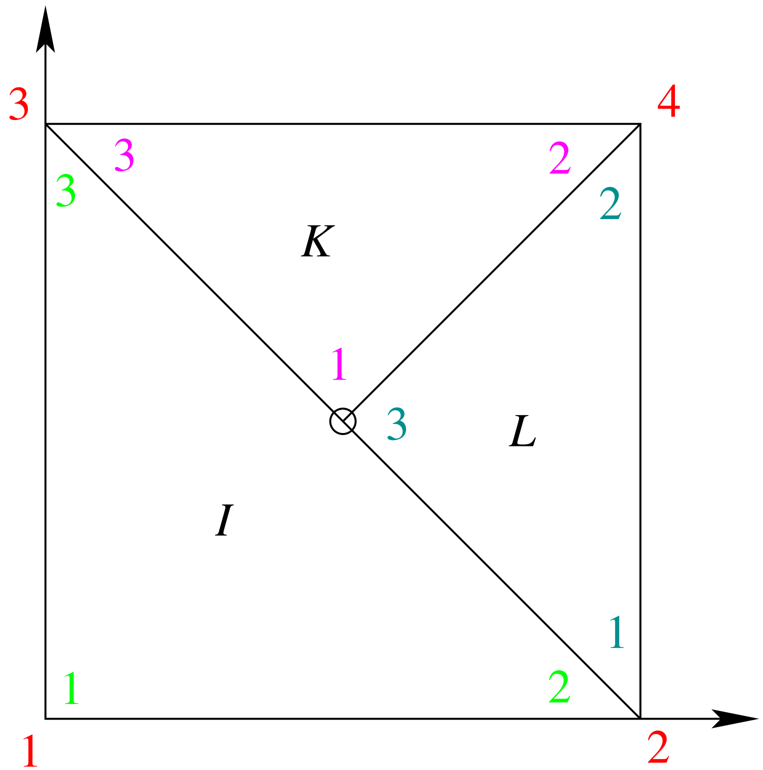

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$$\boldsymbol{T}_K = \begin{pmatrix} & 1 & 2 & 3 & 4 \\ 1 & 0 & 1/2 & 1/2 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$\Rightarrow$ continuous basis functions.

Generation of T Matrices

- **Regular Mesh:** Counting and assigning indices with respect to topological entities such as vertices, edges and faces. Explained in detail later.

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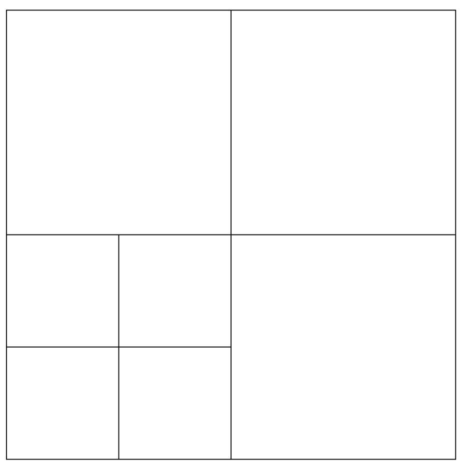
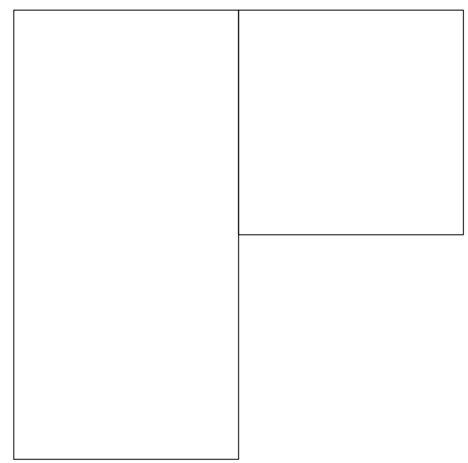
- **Regular Mesh:** Counting and assigning indices with respect to topological entities such as vertices, edges and faces. Explained in detail later.
- **Irregular Mesh:** Irregularity due to a refinement of an initially regular mesh.

OK: , not OK: 

Explanation follows.

T Matrices for Irregular Meshes

Irregularity due to a refinement of an initially regular mesh.

Mesh	$\mathcal{M}$	refine	$\mathcal{M}'$
Basis fcts.	$B = B_{\text{repl}} \cup B_{\text{keep}}$	$\longrightarrow$	$B' = B_{\text{ins}} \cup B_{\text{keep}}$

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B_{repl}: basis fcts. which can be solely described by elements of $\mathcal{M}' \backslash \mathcal{M}$

B_{ins}: basis fcts. generated by regular parts of $\mathcal{M}' \backslash \mathcal{M}$

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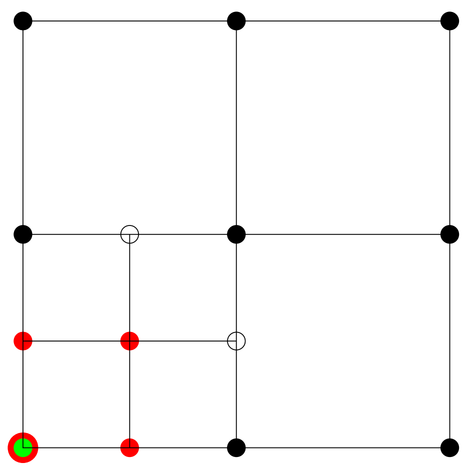

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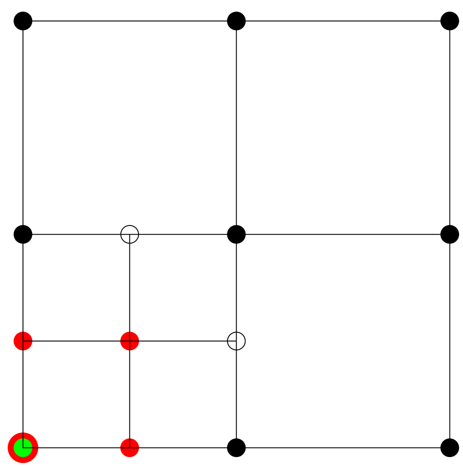
B_{ins}: basis fcts. generated by regular parts of $\mathcal{M}'\backslash\mathcal{M}$

T Matrices for Irregular Meshes

Irregularity due to a refinement of an initially regular mesh.

Mesh	$\mathcal{M}$	refine	$\mathcal{M}'$
Basis fcts.	$B = B_{\mathrm{repl}} \cup B_{\mathrm{keep}}$	$\longrightarrow$	$B' = B_{\mathrm{ins}} \cup B_{\mathrm{keep}}$



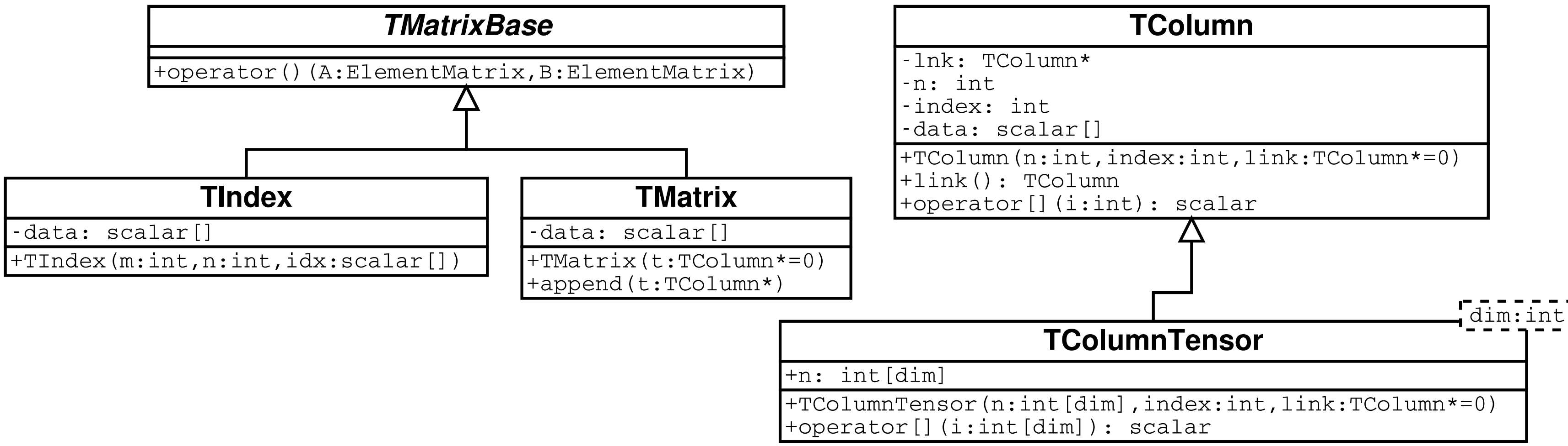
B_{repl}: basis fcts. which can be solely described by elements of $\mathcal{M}'\backslash\mathcal{M}$

B_{ins}: basis fcts. generated by regular parts of $\mathcal{M}'\backslash\mathcal{M}$

Every element of B has a column in the T matrix. Generation is

- easy for B_{ins} (like regular mesh),
- simple for B_{keep}: modify column from $\mathcal{M}$ by S matrix.

Classes for T Matrices



- Different implementations for a T Matrix: TIndex and TMatrix
- TColumn represents a column of a T matrix: coefficients of a global degree of freedom in a particular element
- TColumnTensor: different interface to the data of TColumn with multi-indices

Overview

- Introduction: FEM & Exponential Convergence
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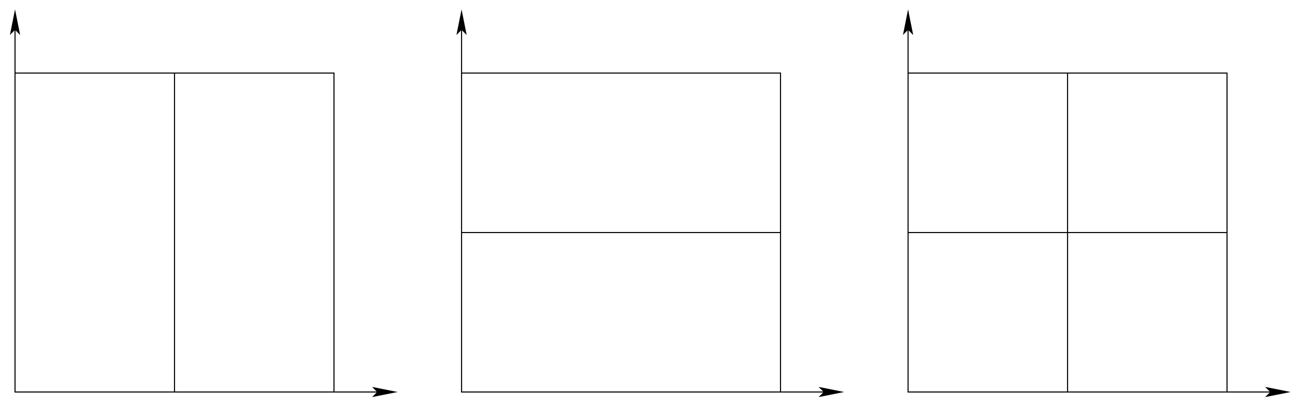
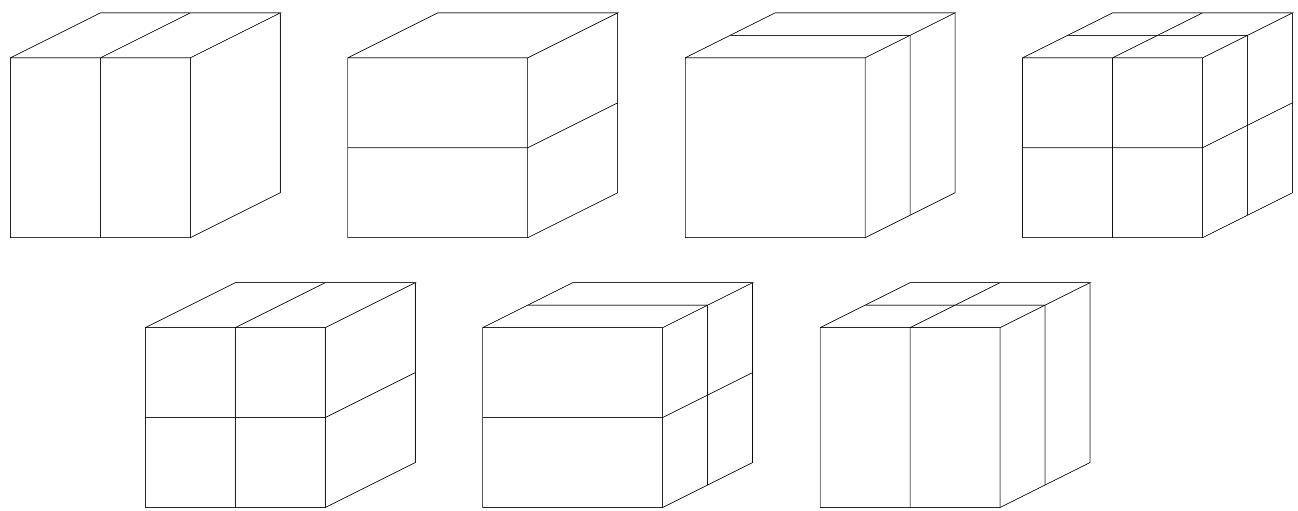
Subdivisions

Subdivisions of a quadrilateral in 2D:



Subdivisions of a hexahedron in 3D:



S Matrix

Definition 2 (S Matrix). *Let $K' \subset K$ be the result of a refinement of element K. The S matrix $\boldsymbol{S}_{K'K} \in \mathbb{R}^{m_{K'} \times m_K}$ is defined by*

$$\phi_j^K\Big|_{K'} = \sum_{l=1}^{m_{K'}} [\boldsymbol{S}_{K'K}]_{lj}\, \phi_l^{K'}$$

as vectors:

$$\underline{\phi}^K\Big|_{K'} = \boldsymbol{S}_{K'K}^{\top} \underline{\phi}^{K'}$$

$\phi_j^K\big|_{K'}$ is represented as a linear combination of the shape functions $\left\{\phi_l^{K'}\right\}_{l=1}^{m_{K'}}$ of K'.

Application of S Matrix

Proposition 1. *Let $K' \subset K$ be the result of a refinement of an element K. Then, the T matrix of K' can be computed as*

$$\boldsymbol{T}_{K'} = \boldsymbol{S}_{K'K}\boldsymbol{T}_K^{\mathrm{keep}} + \boldsymbol{T}_{K'}^{\mathrm{ins}}$$

where $\boldsymbol{T}_K^{\mathrm{keep}}$ denotes the T matrix of element K (with columns not related to functions in B_{keep} set to zero) and $\boldsymbol{T}_{K'}^{\mathrm{ins}}$ the T matrix for functions in B_{ins} with respect to K'.

Proposition 2. *Let $\hat{K}' \subset \hat{K}$ be the result of a refinement of the reference element $\hat{K}$ with $H : \hat{K} \to \hat{K}'$ the subdivision map. The element maps are*

$$F_K : \hat{K} \to K \text{ and } F_{K'} : \hat{K} \to K'$$

and $F_{K'} \circ H^{-1} = F_K$ holds. Then, $\boldsymbol{S}_{\hat{K}'\hat{K}} = \boldsymbol{S}_{K'K}$.

S Matrix in Dimension $d = 1$

Subdividing $\hat{J} = (0,1)$ in $\hat{J}' = (0, 1/2)$ and $\hat{J}^\star = (1/2, 1)$ with the reference element shape functions

$$N_j(\xi) = \begin{cases} 1-\xi & j = 1 \\ \xi & j = 2 \\ \xi(1-\xi)P^{1,1}_{j-3}(2\xi - 1) & j = 3, \dots, J \end{cases}$$

yields (solving a linear system) for $J = 4$:

$$\boldsymbol{S}_{\hat{J}',\hat{J}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 1/4 & 0 \\ 0 & 0 & 1/4 & -3/4 \\ 0 & 0 & 0 & 1/8 \end{pmatrix} \text{ and } \boldsymbol{S}_{\hat{J}^\star,\hat{J}} = \begin{pmatrix} 1/2 & 1/2 & 1/4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 1/8 \end{pmatrix}.$$

Hierarchic shape functions $\Rightarrow$ hierarchic S matrices.

S Matrices: Tensor Product in 2D – I

- $d > 1$ with hexahedral meshes $\Rightarrow$ S matrices are built from tensor products of 1D S matrices.

S Matrices: Tensor Product in 2D – I

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- In 2D: $N_{i,j} = N_i \otimes N_j$, the four bilinear shape functions are:

$$N_{1,2}(\underline{\xi}) = N_1(\xi_1) \cdot N_2(\xi_2) \qquad N_{2,2}(\underline{\xi}) = N_2(\xi_1) \cdot N_2(\xi_2)$$

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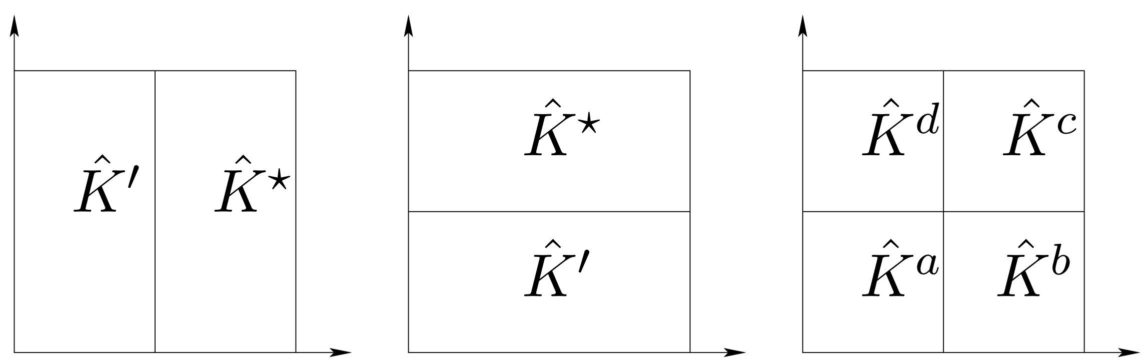
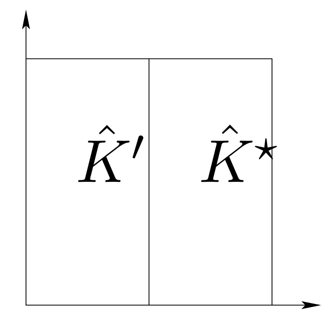
- Consider the subdivisions:



S Matrices: Tensor Product in 2D – II

Subdivision map of left variant: $H : \hat{K} \to \hat{K}', \underline{\xi} \mapsto \begin{pmatrix} \xi_1/2 \\ \xi_2 \end{pmatrix}$. S matrix $\boldsymbol{S}_{\hat{K}'\hat{K}}$ is defined by:

$$N_{i,j}|_{\hat{K}'} = \sum_{k,l} \left[\boldsymbol{S}_{\hat{K}'\hat{K}}\right]_{(k,l),(i,j)} N_{k,l} \circ H^{-1}.$$



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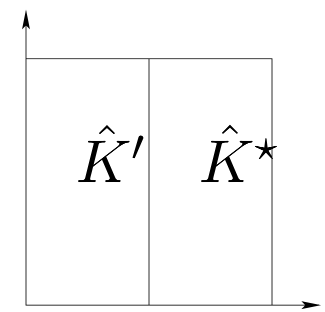


Tensor product shape functions:

$$(N_i \otimes N_j)|_{\hat{K}'} = \sum_{k,l} \left[\boldsymbol{S}_{\hat{K}'\hat{K}}\right]_{(k,l),(i,j)} (N_k \otimes N_l) \circ H^{-1}. \quad (1)$$

S Matrices: Tensor Product in 2D – III

S matrices for 1D reference element shape fcts. used in (1):

$$N_i|_{\hat{\jmath}'} = \sum_m [\boldsymbol{S}_{\hat{\jmath}',\hat{\jmath}}]_{mi} N_m \circ G^{-1}$$ for the ξ_1 part and

$$N_j = \sum_n [\boldsymbol{E}]_{nj} N_n$$ for the ξ_2 part,

where $G : \xi \mapsto \xi/2$.

S Matrices: Tensor Product in 2D – III

S matrices for 1D reference element shape fcts. used in (1):

$$N_i|_{\hat{j}'} = \sum_m \left[\boldsymbol{S}_{\hat{j}',\hat{j}}\right]_{mi} N_m \circ G^{-1}$$ for the ξ_1 part and

$$N_j = \sum_n [\boldsymbol{E}]_{nj} N_n$$ for the ξ_2 part,

where $G : \xi \mapsto \xi/2$. Plugging into the left hand side of (1) yields:

$$\begin{aligned}(N_i \otimes N_j)|_{\hat{K}'} = N_i|_{\hat{j}'} \otimes N_j &= \sum_{m,n} \left(\left[\boldsymbol{S}_{\hat{j}',\hat{j}}\right]_{mi} N_m \circ G^{-1}\right) \otimes \left([\boldsymbol{E}]_{nj} N_n\right) \\ &= \sum_{m,n} \left[\boldsymbol{S}_{\hat{j}',\hat{j}}\right]_{mi} \cdot [\boldsymbol{E}]_{nj} N_m \circ G^{-1} \otimes N_n .\end{aligned}$$

S Matrices: Tensor Product in 2D – IV

Comparing with the right hand side of (1):

$$\sum_{m,n} \left[\boldsymbol{S}_{\hat{J}',\hat{J}}\right]_{mi} \cdot [\boldsymbol{E}]_{nj}\, N_m \circ G^{-1} \otimes N_n$$

$$= \sum_{k,l} \left[\boldsymbol{S}_{\hat{K}'\hat{K}}\right]_{(k,l),(i,j)} N_k \circ G^{-1} \otimes N_l.$$

S Matrices: Tensor Product in 2D – IV

Comparing with the right hand side of (1):

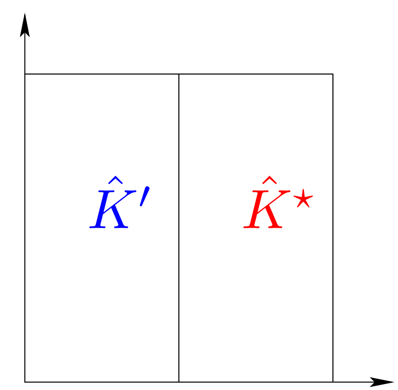
$$\sum_{m,n} [\boldsymbol{S}_{\hat{J}'\hat{J}}]_{mi} \cdot [\boldsymbol{E}]_{nj} \, N_m \circ G^{-1} \otimes N_n$$

$$= \sum_{k,l} [\boldsymbol{S}_{\hat{K}'\hat{K}}]_{(k,l),(i,j)} \, N_k \circ G^{-1} \otimes N_l .$$

Therefore for the vertical subdivision:

$\boldsymbol{S}_{\hat{K}'\hat{K}} = \boldsymbol{S}_{\hat{J}'\hat{J}} \otimes \boldsymbol{E}$ for the left quad $\hat{K}'$,

$\boldsymbol{S}_{\hat{K}^\star\hat{K}} = \boldsymbol{S}_{\hat{J}^\star\hat{J}} \otimes \boldsymbol{E}$ for the right quad $\hat{K}^\star$.

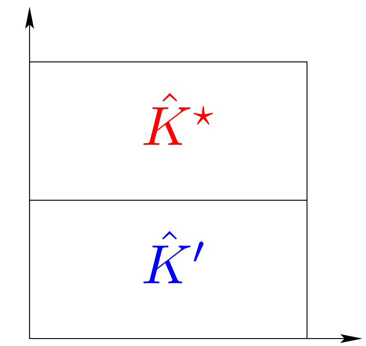


S Matrices: Tensor Product in 2D – V

Horizontal subdivision:

$\boldsymbol{S}_{\hat{K}'\hat{K}} = \boldsymbol{E} \otimes \boldsymbol{S}_{\hat{J}'\hat{J}}$ for the bottom quad $\hat{K}'$,

$\boldsymbol{S}_{\hat{K}^\star\hat{K}} = \boldsymbol{E} \otimes \boldsymbol{S}_{\hat{J}^\star\hat{J}}$ for the top quad $\hat{K}^\star$.

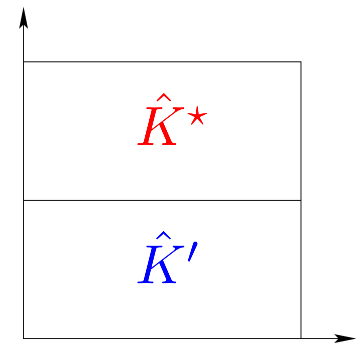


S Matrices: Tensor Product in 2D – V

Horizontal subdivision:

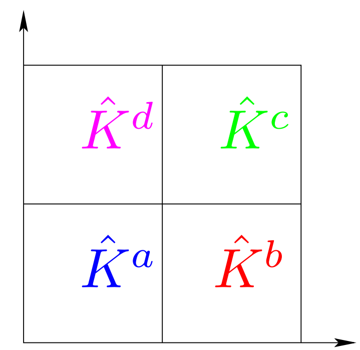
$$\boldsymbol{S}_{\hat{K}'\hat{K}} = \boldsymbol{E} \otimes \boldsymbol{S}_{\hat{J}'\hat{J}} \qquad \text{for the bottom quad } \hat{K}',$$

$$\boldsymbol{S}_{\hat{K}^\star\hat{K}} = \boldsymbol{E} \otimes \boldsymbol{S}_{\hat{J}^\star\hat{J}} \qquad \text{for the top quad } \hat{K}^\star.$$



Subdivision into four quads:

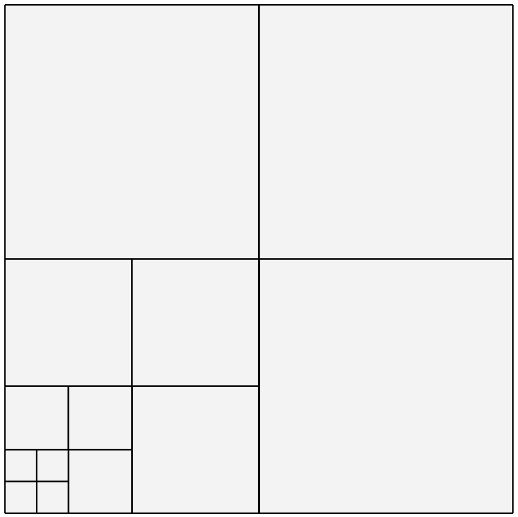
- subdivide $\hat{K}$ horizontally into two children
- subdivide upper and lower
  child vertically into $\hat{K}^d$ and $\hat{K}^c$ and $\hat{K}^a$ and $\hat{K}^b$ resp.



$$\boldsymbol{S}_{\hat{K}^d\hat{K}} = \left(\boldsymbol{S}_{\hat{J}'\hat{J}} \otimes \boldsymbol{E}\right) \cdot \left(\boldsymbol{E} \otimes \boldsymbol{S}_{\hat{J}^\star\hat{J}}\right) \qquad \boldsymbol{S}_{\hat{K}^c\hat{K}} = \left(\boldsymbol{S}_{\hat{J}^\star\hat{J}} \otimes \boldsymbol{E}\right) \cdot \left(\boldsymbol{E} \otimes \boldsymbol{S}_{\hat{J}^\star\hat{J}}\right)$$

$$\boldsymbol{S}_{\hat{K}^a\hat{K}} = \left(\boldsymbol{S}_{\hat{J}'\hat{J}} \otimes \boldsymbol{E}\right) \cdot \left(\boldsymbol{E} \otimes \boldsymbol{S}_{\hat{J}'\hat{J}}\right) \qquad \boldsymbol{S}_{\hat{K}^b\hat{K}} = \left(\boldsymbol{S}_{\hat{J}^\star\hat{J}} \otimes \boldsymbol{E}\right) \cdot \left(\boldsymbol{E} \otimes \boldsymbol{S}_{\hat{J}'\hat{J}}\right)$$

Meshes



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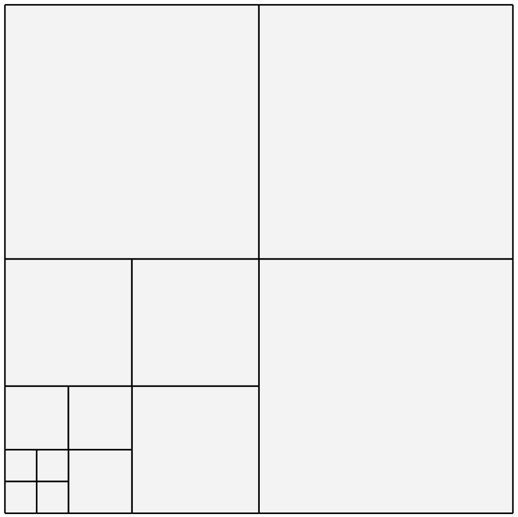

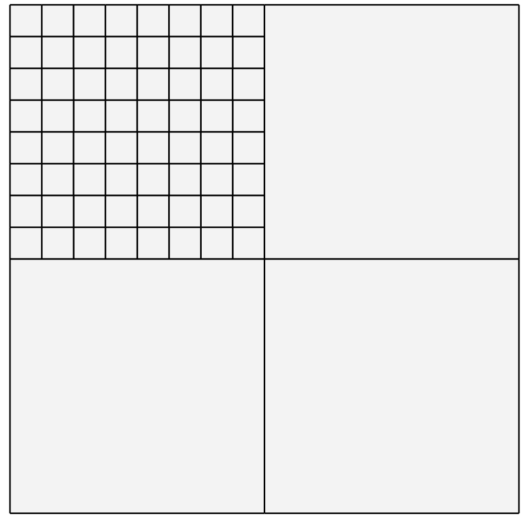

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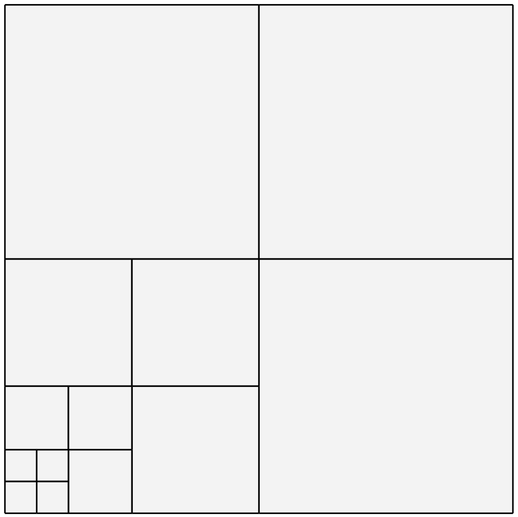
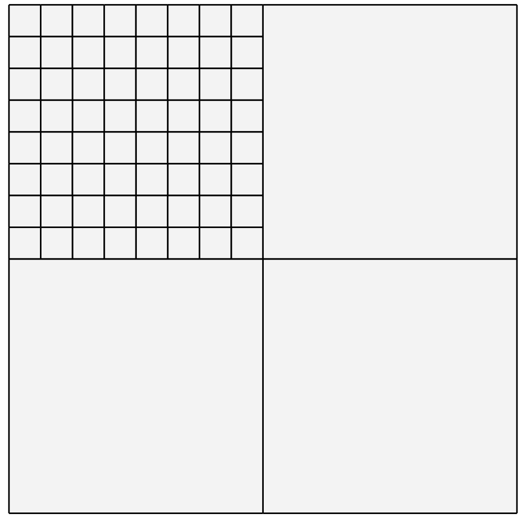
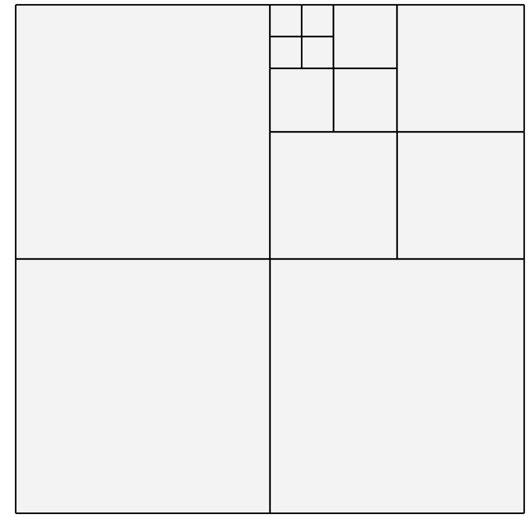




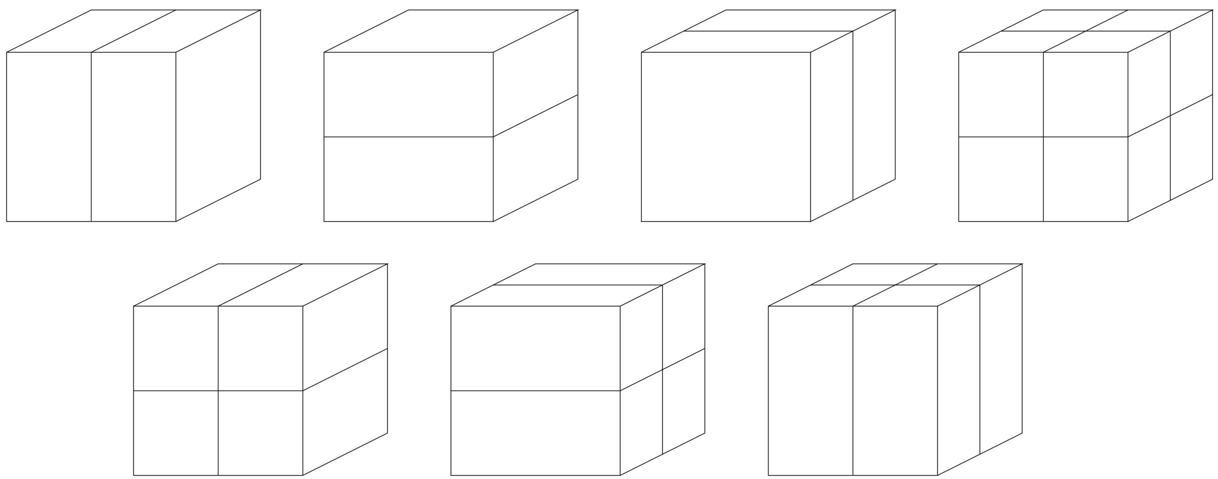


S Matrices: Tensor-Product in 3D

Same idea as in 2D, just of this form:

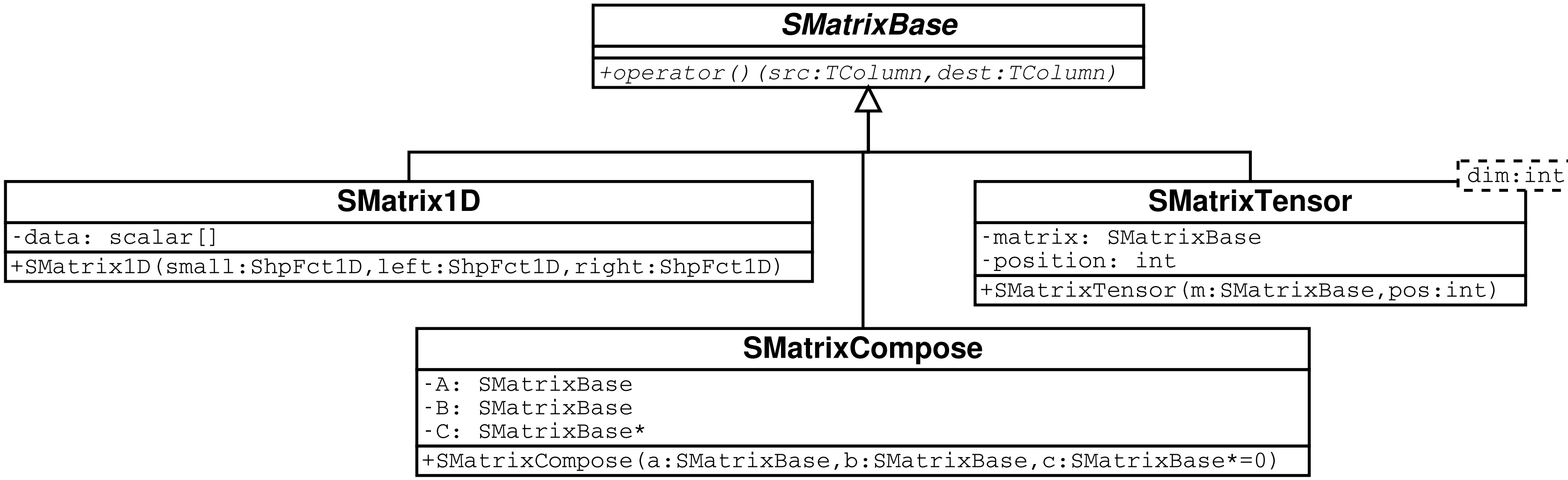
$$S_{\hat{K}'\hat{K}} = \prod (A \otimes B \otimes C)$$

in each of the factors, one of A, B or C is an 1D S matrix.
Depending on the factors, 7 subdivisions are possible:



Concepts: arbitrary number and combination of these 7 subdivisions in 3D.

Classes for S Matrices



- SMatrix1D contains the coefficients, all other classes make use of that
- SMatrixCompose implements $\cdot$
- SMatrixTensor implements $\otimes$

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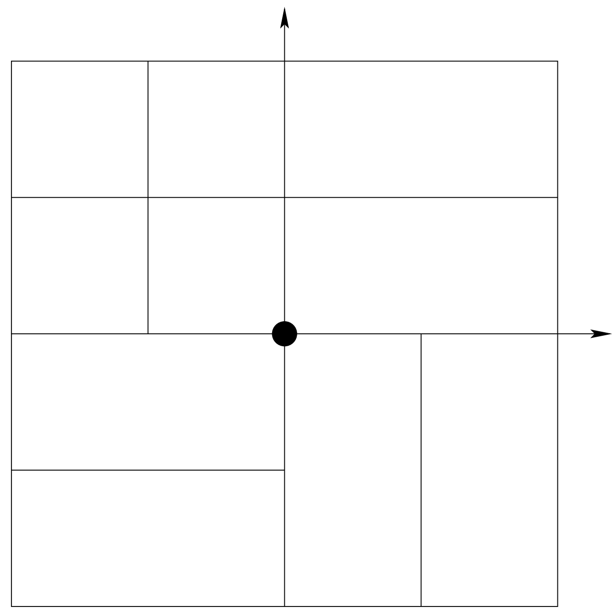
Anisotropic and Conforming

Main point: find the cells (either coarse or fine) which belong to the support of a certain basis function.



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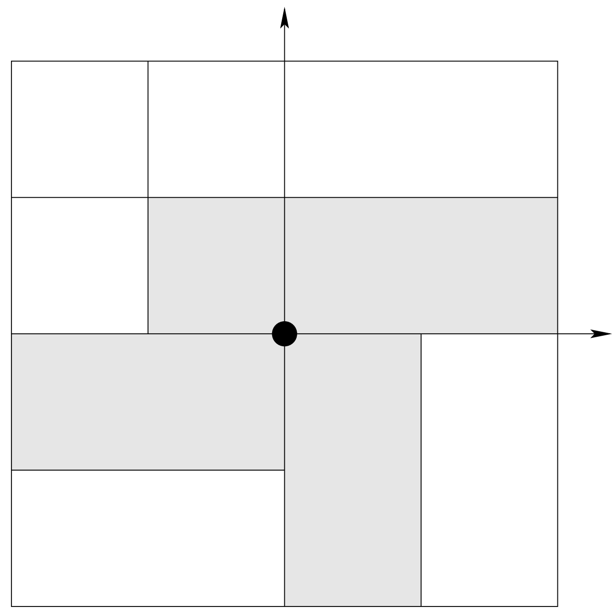

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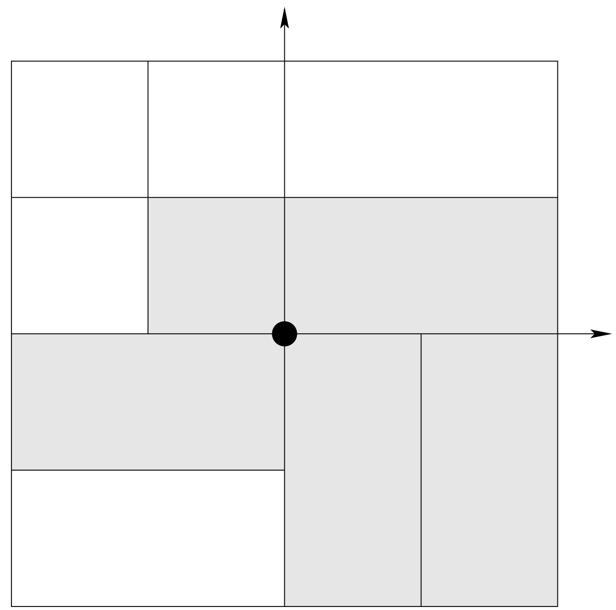

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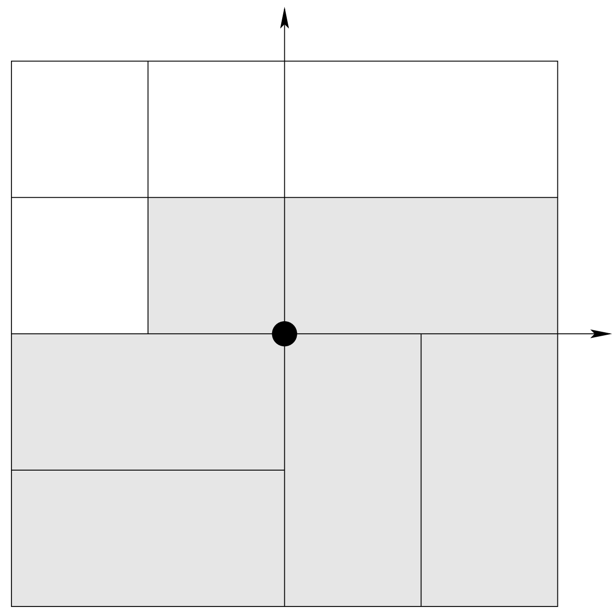

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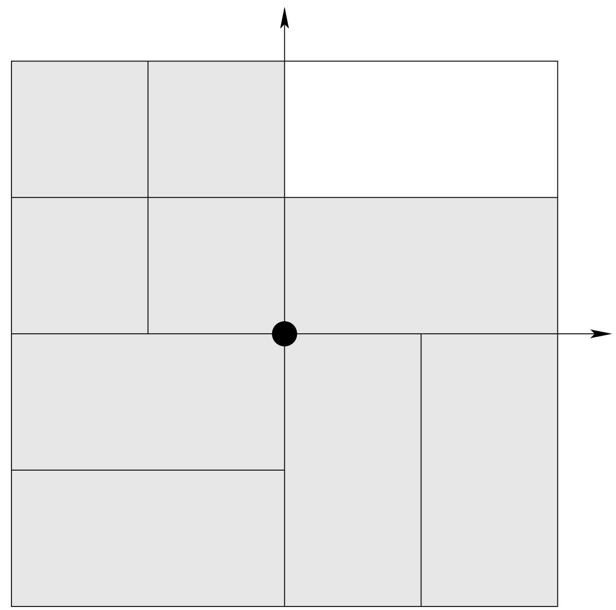

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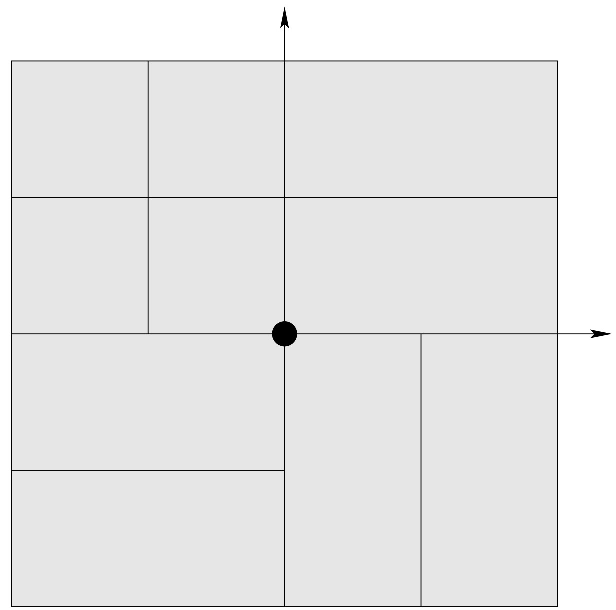

Condition for Continuity

- In order to have continuous global basis functions Φ_i, the unisolvent sets on the interfaces in the support of Φ_i have to match.

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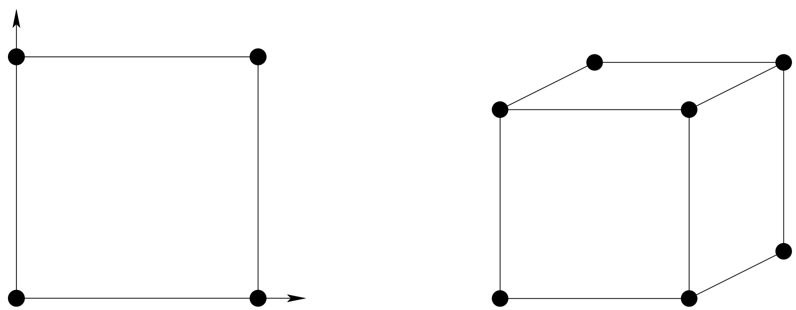

$\Rightarrow$ matching edges which coincide in a vertex is sufficient

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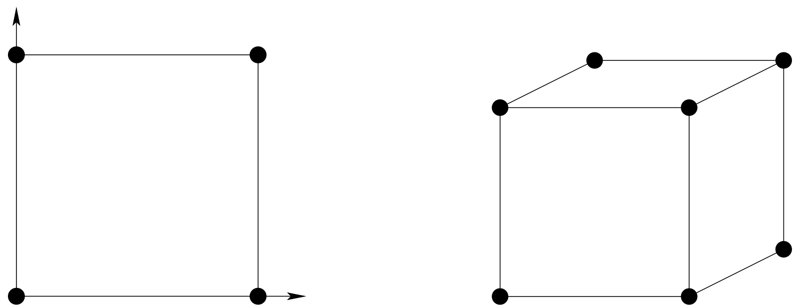


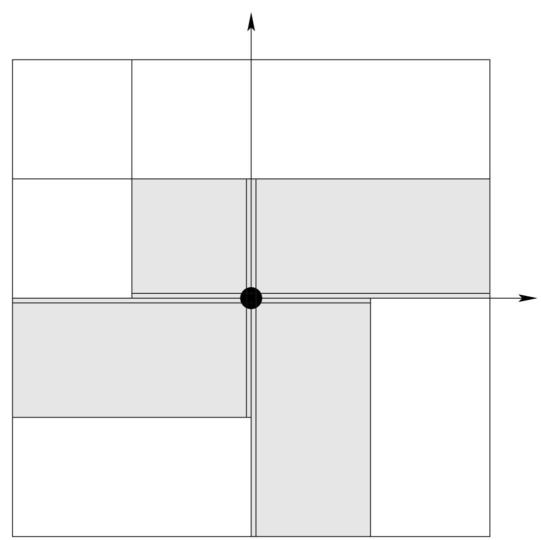
$\Rightarrow$ matching edges which coincide in a vertex is sufficient

- Unisolvent set for $\mathcal{Q}_p$ are
  - additional $p-1$ points on every edge
  - additional $(p-1)^2$ points on every face
  - additional $(p-1)^3$ points in the interior

$\Rightarrow$ matching faces which coincide in an edge is sufficient for continuous edge modes

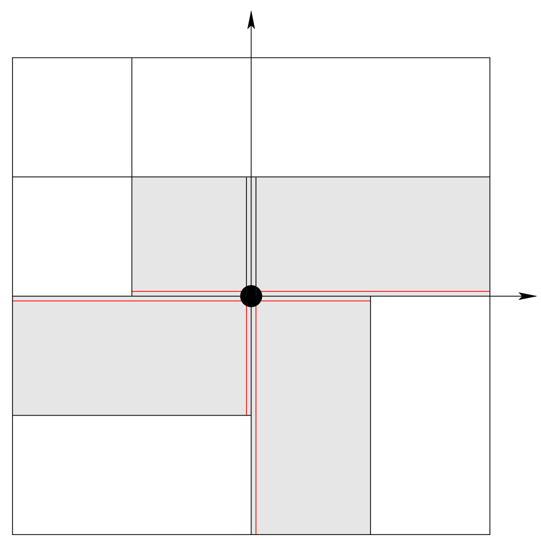
Algorithm for Continuity (Vertex)

- In every cell of the finest mesh, register all edges and cells in their vertices.



Algorithm for Continuity (Vertex)

- In every cell of the finest mesh, register all edges and cells in their vertices.
- For every vertex, while something changed in the last loop:
  - Check if some of the edges of the vertex have a relationship (ancestor / descendant).



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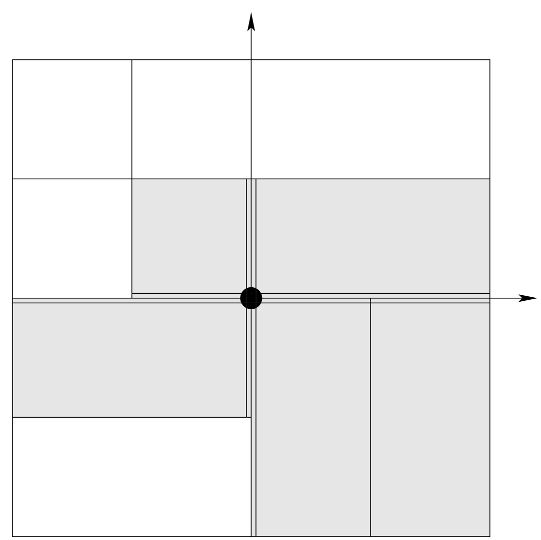
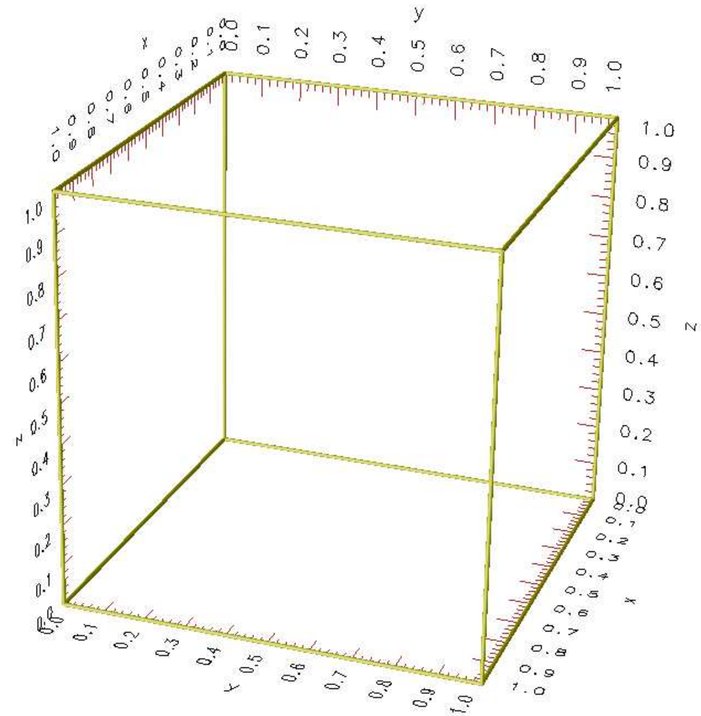
- For every vertex, while something changed in the last loop:
  - Check if some of the edges of the vertex have a relationship (ancestor / descendant).
  - If two edges are related, exchange the smaller cell in the list of the vertex by the cell matching the larger cell.
  - Delete the list of edges and rebuild it from the list of cells.



Sample Basis Functions



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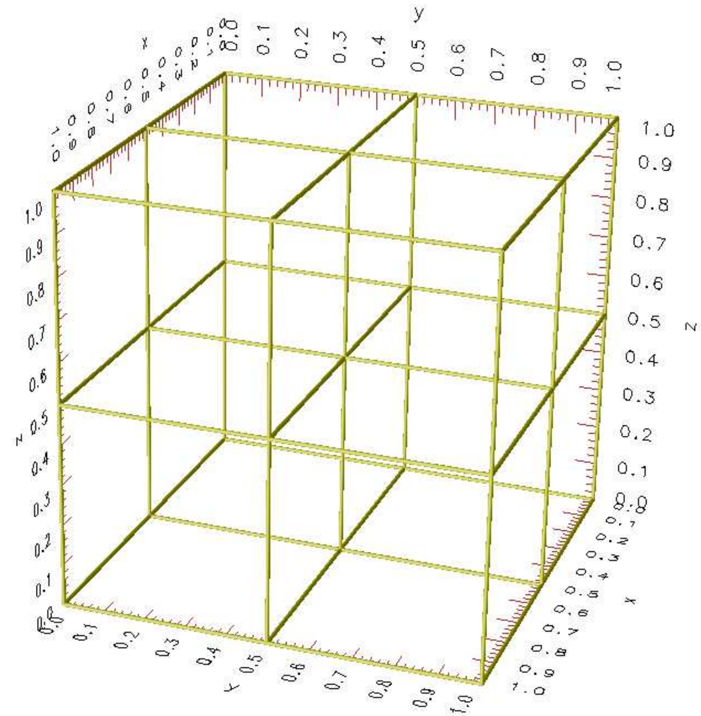

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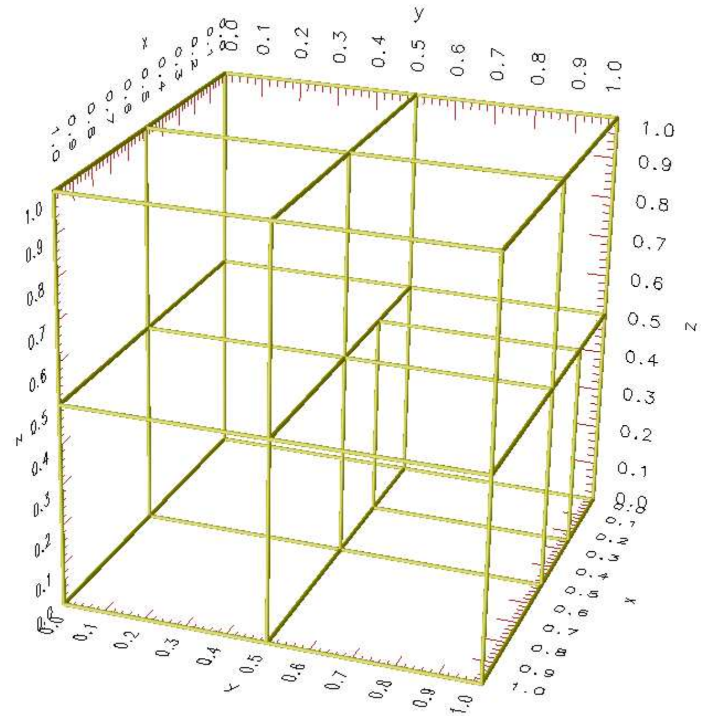

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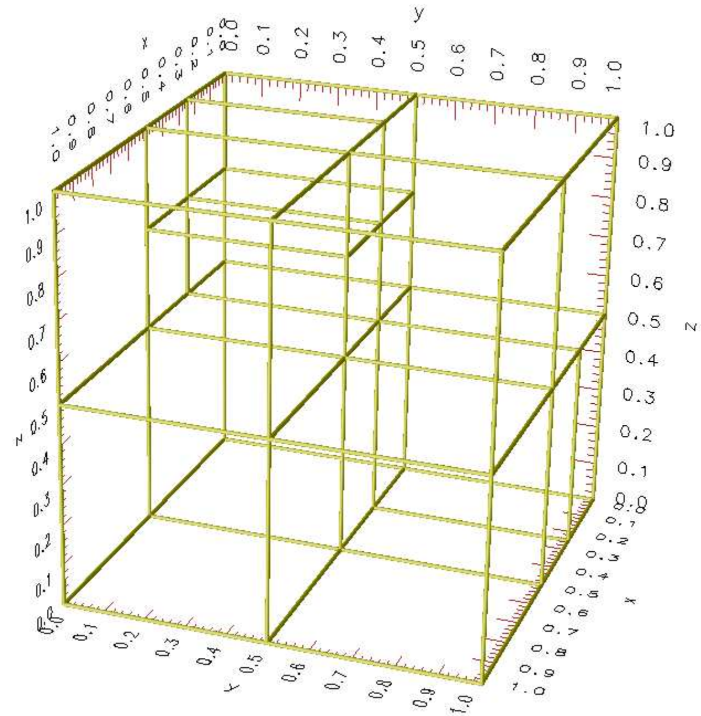
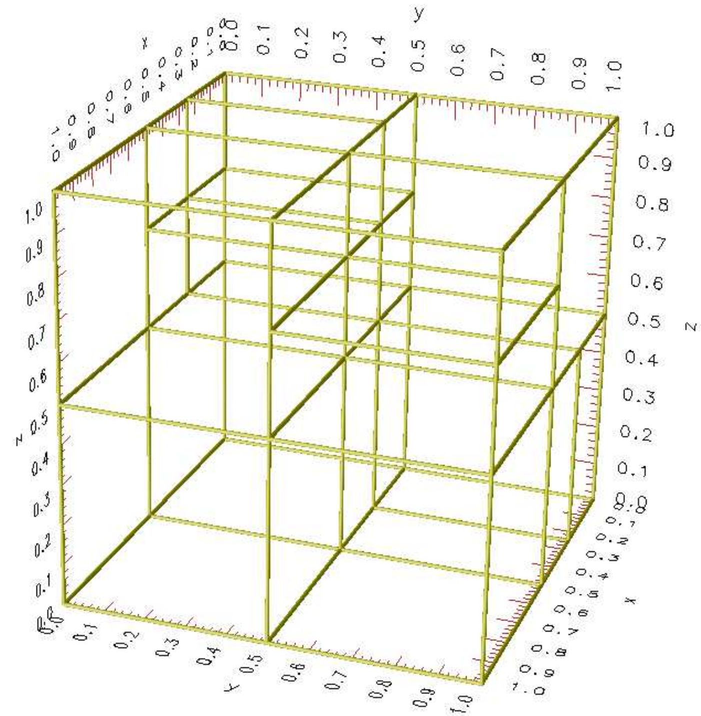
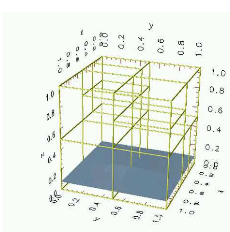
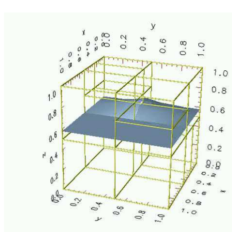
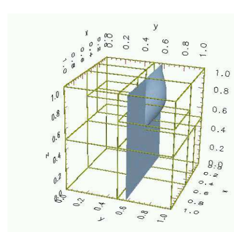


Sample Basis Functions

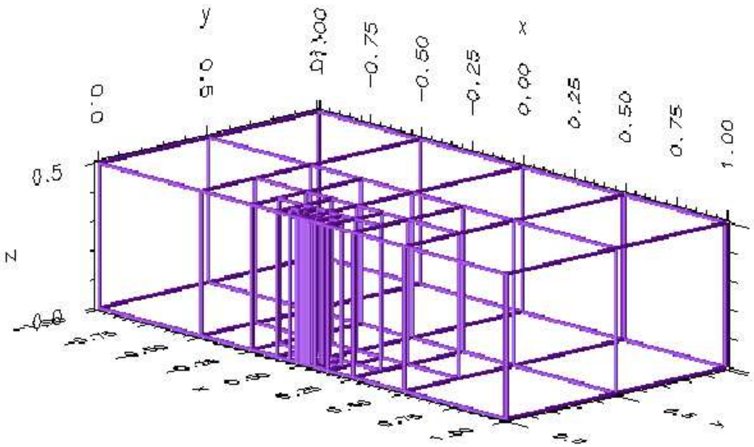


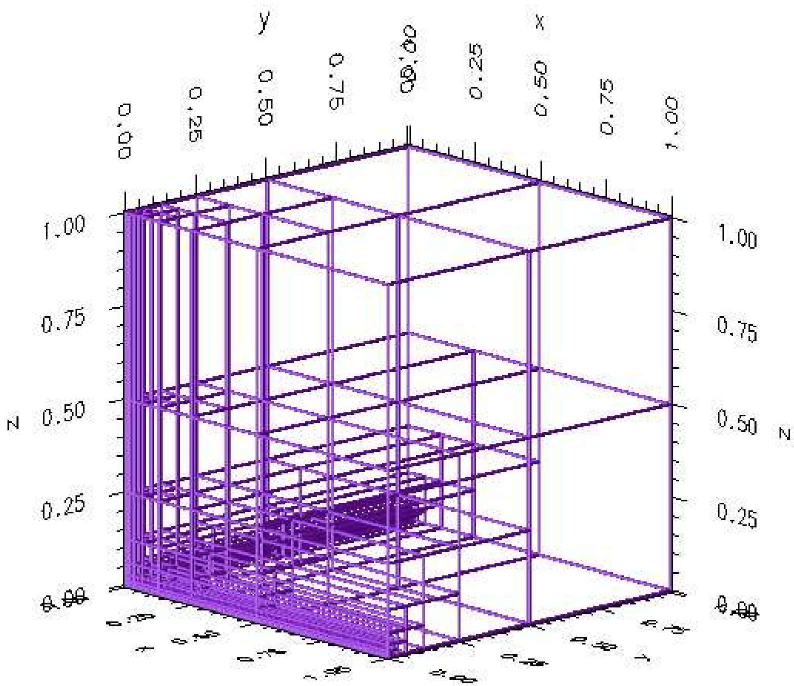


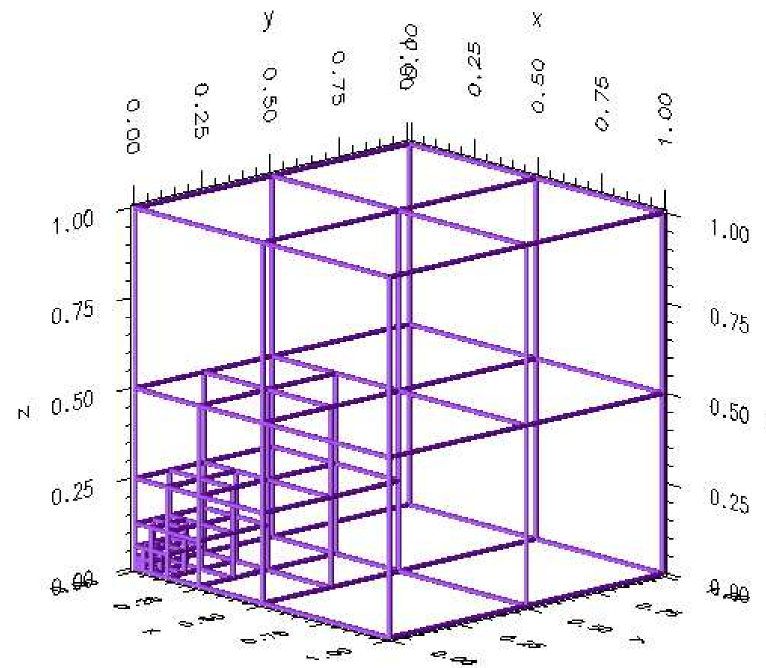




Sample hp ***Meshes***

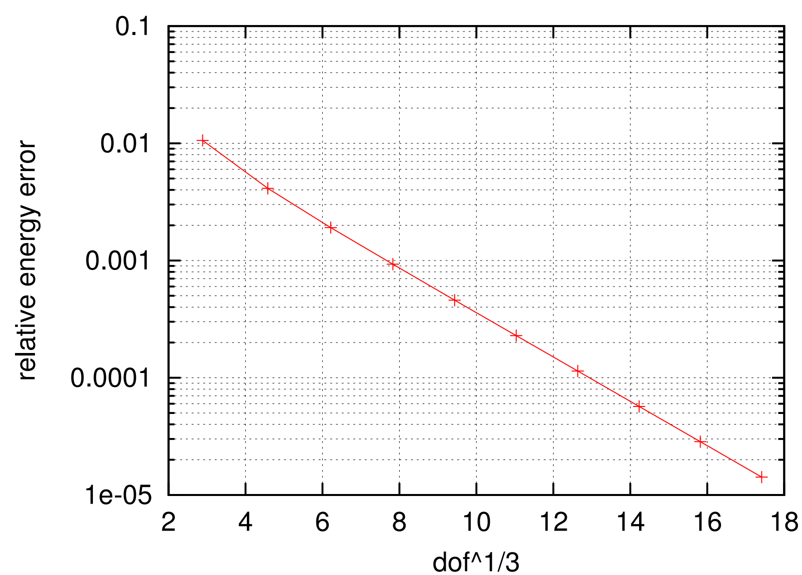


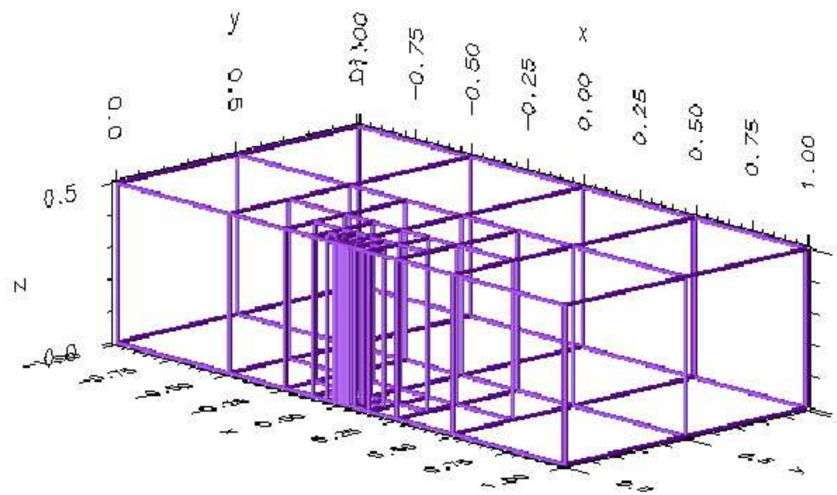




Scalar Computations: Pseudo 3D

Edge type singularity.





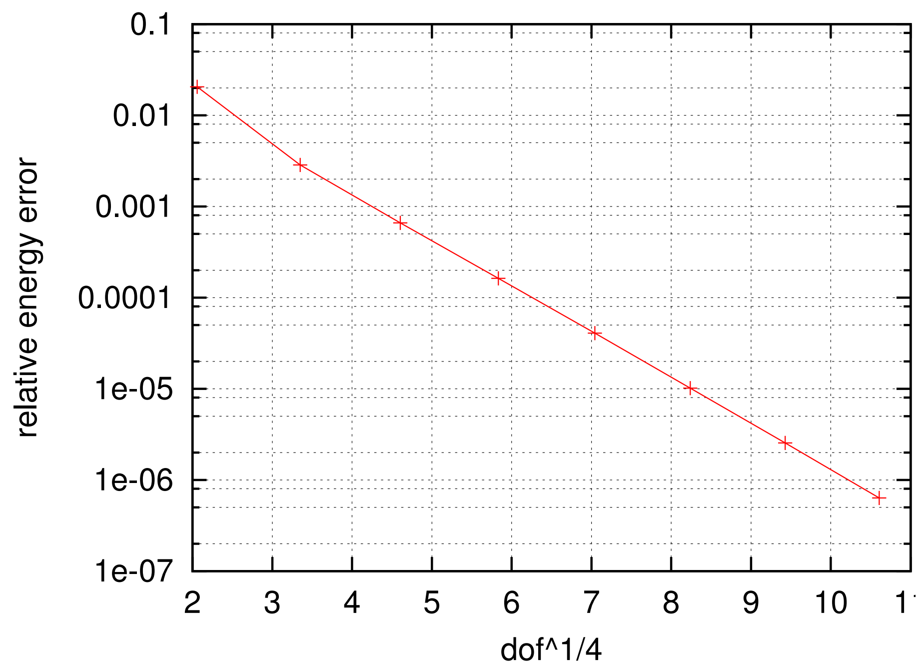
$$-\Delta u + u = f \text{ in } \Omega = (-1,1) \times (0,1) \times (0, 1/2)$$

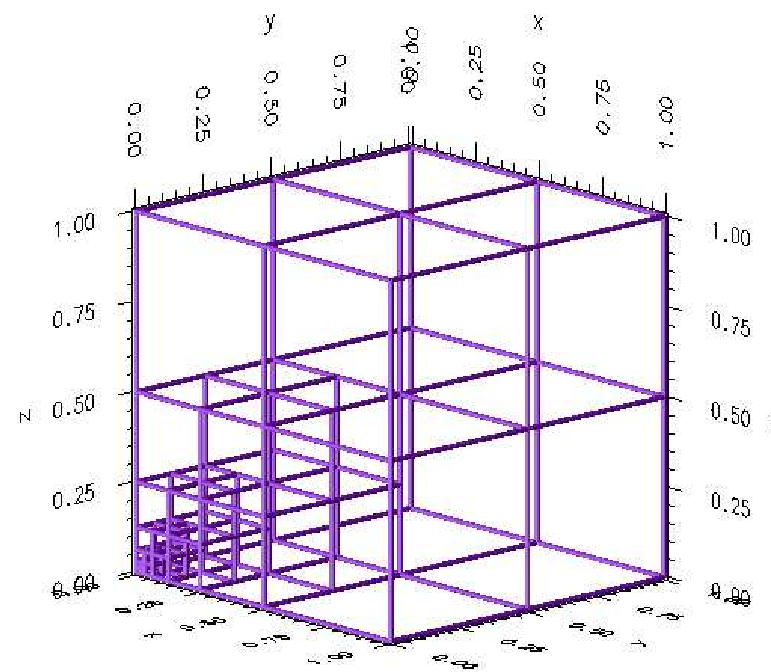
$$u(r,\phi,z) = \sqrt{r}\sin(\phi/2)z(1-z) \quad \text{in } \Omega$$

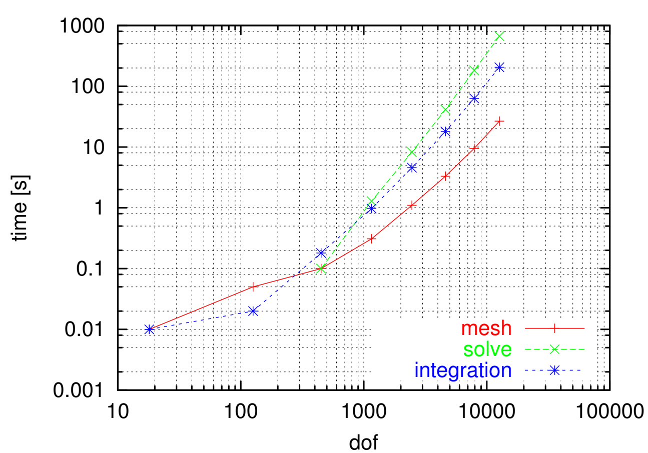
$$u = 0 \quad \text{on } \{z=0\} \subset \partial\Omega$$

$$\text{and on } \{y=0\} \cap \{x \geq 0\} \subset \partial\Omega$$

Scalar Computations: Vertex Singularity







Vertex type singularity.

$$-\Delta u + u = f \text{ in } \Omega = (0,1)^3$$

$$u(r,\theta,\phi) = \sqrt{r}\sin\theta\sin\phi \qquad \text{in } \Omega$$

$$u = 0 \qquad \text{on } \{y=0\} \subset \partial\Omega$$

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Overview

- Introduction: FEM & Exponential Convergence
- Assembling
- Handling Hanging Nodes
- Finding Regular Supports
- Maxwell Eigenvalue Problems
- Perspectives

Eigenvalue Problems

Source problem:

Find u with $-\Delta u + cu = f$ in Ω and $u = 0$ on $\partial\Omega$.

A variational form ($\cdot v$, $\int$, integration by parts):

Find $u \in V$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx + c \int_\Omega uv \, dx = \int_\Omega fv \, dx \qquad \forall v \in V$$

$$\Rightarrow \quad (\boldsymbol{A} + c\boldsymbol{M})\underline{u}_N = \underline{l}_N$$ solve for $\underline{u}_N \in V_N$.

Eigenvalue Problems

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$$\Rightarrow \quad (\boldsymbol{A} + c\boldsymbol{M})\underline{u}_N = \underline{l}_N \qquad \text{solve for } \underline{u}_N \in V_N.$$

Eigenvalue Problem:
Find an Eigenpair (λ, u) with $-\Delta u = \lambda u$ in Ω and $u = 0$ on $\partial\Omega$.
Find $(\lambda, u) \in \mathbb{R} \times V$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \lambda \int_\Omega uv \, dx \qquad \forall v \in V$$

$$\Rightarrow \quad \boldsymbol{A}\underline{u}_N = \lambda_N \boldsymbol{M}\underline{u}_N \qquad \text{solve for } (\lambda_N, \underline{u}_N) \in \mathbb{R} \times V_N.$$

Convergence of Eigenvalues

Eigenvectors:

$$\|w_m - w_{m,N}\|_{H^1} \leq C \sup_{v \in W_m} \inf_{v_N \in V_N} \|v - v_N\|_{H^1}$$

Simple Eigenvalues:

$$|\lambda_m - \lambda_{m,N}| \leq C \sup_{v \in W_m} \inf_{v_N \in V_N} \|v - v_N\|_{H^1}^2$$

Eigenvalues converge twice as fast as Eigenvectors.
Eigenvectors converge quasi-optimally.

For $\|v - v_N\|_{H^1}$, exponential convergence is possible.

Maxwell Equations

$$-\partial_t \underline{D} + \operatorname{curl} \underline{H} = \sigma \underline{E} + \underline{j} \qquad \text{Ampère's Law} \qquad (2)$$

$$\partial_t \underline{B} + \operatorname{curl} \underline{E} = 0 \qquad \text{Farraday's Law} \qquad (3)$$

$\underline{j}$: current density, $\underline{H}, \underline{E}$: magnetic & electric field, $\underline{B}, \underline{D}$: magnetic & electric induction.
Constitutive laws (in general: $\boldsymbol{\varepsilon}, \boldsymbol{\mu} \in \mathbb{R}^{3\times 3}$ later assumed to be scalars):
$\underline{D} = \boldsymbol{\varepsilon}\underline{E}$ and $\underline{B} = \boldsymbol{\mu}\underline{H}$ applied to (2) & (3):

$$-\partial_t \varepsilon \underline{E} + \operatorname{curl} \underline{H} = \sigma \underline{E} + \underline{j} \qquad \partial_t \mu \underline{H} + \operatorname{curl} \underline{E} = 0$$

Consider time harmonic solutions, Ansatz:

$$\underline{E}(\underline{x}, t) = \operatorname{Re}\big(\underline{E}(\underline{x}) e^{i\omega t}\big) \qquad \underline{H}(\underline{x}, t) = \operatorname{Re}\big(\underline{H}(\underline{x}) e^{i\omega t}\big)$$

ie., combinations of $\sin$ and $\cos$.

Towards a Variational Form

Time harmonic Maxwell equations:

$$-i\omega\varepsilon\underline{E} + \operatorname{curl}\underline{H} = \sigma\underline{E} + \underline{j} \qquad i\omega\mu\underline{H} + \operatorname{curl}\underline{E} = 0$$

$$\operatorname{curl}(\varepsilon^{-1}\operatorname{curl}\underline{H}) - \omega^2\tilde{\mu}\underline{H} = \operatorname{curl}(\varepsilon^{-1}\underline{j}) \qquad \operatorname{curl}(\mu^{-1}\operatorname{curl}\underline{E}) - \omega^2\tilde{\varepsilon}\underline{E} = -i\omega\underline{j}$$

with perfect conductor boundary conditions ($\sigma \to \infty$):

$$\mu\underline{H}\cdot\underline{n} = 0 \qquad \underline{E}\wedge\underline{n} = 0$$

Spaces for these equations:

$$H(\operatorname{div},\Omega) := \left\{\underline{u} \in L^2(\Omega)^3 : \operatorname{div}\underline{u} \in L^2(\Omega)\right\}$$

$$H(\operatorname{curl},\Omega) := \left\{\underline{u} \in L^2(\Omega)^3 : \operatorname{curl}\underline{u} \in L^2(\Omega)\right\}$$

Variational Electric Source Problem

Consider

$$\mathrm{curl}(\mu^{-1}\,\mathrm{curl}\,\underline{E}) - \omega^2\tilde{\varepsilon}\underline{E} = -i\omega\underline{j} \text{ in } \Omega \text{ and } \underline{E}\wedge\underline{n} = 0 \text{ on } \partial\Omega.$$

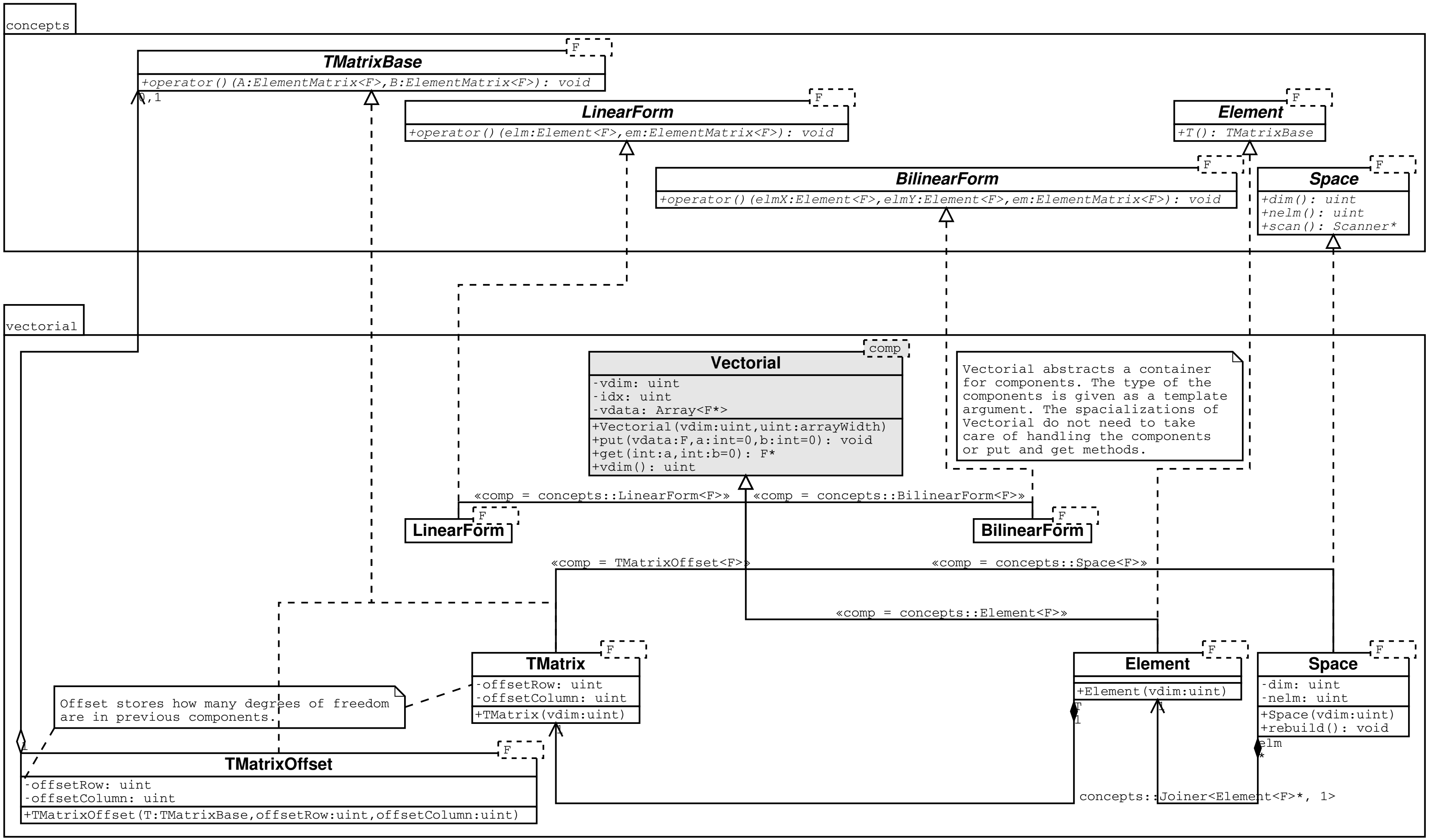
in variational form:
Find $\underline{E} \in H_0(\mathrm{curl},\Omega)$ with $\mathrm{div}\,\varepsilon\underline{E} = 0$ such that

$$\int_\Omega \mu^{-1}\,\mathrm{curl}\,\underline{E}\cdot\mathrm{curl}\,\underline{F} - \omega^2\int_\Omega \varepsilon\underline{E}\cdot\underline{F} = \int_\Omega i\omega\underline{j}\cdot\underline{F} \quad \forall\underline{F}\in H_0(\mathrm{curl},\Omega)$$

Constraint $\mathrm{div}\,\varepsilon\underline{E} = 0$ makes discretisation difficult (Nédelec elements).
Better introduce $X_n := \{\underline{E}\in H_0(\mathrm{curl},\Omega) : \mathrm{div}\,\underline{E}\in L^2(\Omega)\}$ and the variational form
Find $\underline{E}\in X_n$ such that

$$\int_\Omega \mu^{-1}\,\mathrm{curl}\,\underline{E}\cdot\mathrm{curl}\,\underline{F} + \int_\Omega \mathrm{div}\,\underline{E}\,\mathrm{div}\,\underline{F} - \omega^2\int_\Omega \varepsilon\underline{E}\cdot\underline{F} = \int_\Omega i\omega\underline{j}\cdot\underline{F} \quad \forall\underline{F}\in X_n$$

Classes for Vector Valued Problems



Electric Eigenvalue Problem

Find $\omega > 0$ sucht that $\exists \underline{E} \in X_n \setminus \{0\}$ with

$$\int_\Omega \mu^{-1} \operatorname{curl} \underline{E} \cdot \operatorname{curl} \underline{F} + \int_\Omega \operatorname{div} \underline{E} \operatorname{div} \underline{F} = \omega^2 \int_\Omega \varepsilon \underline{E} \cdot \underline{F} \quad \forall \underline{F} \in X_n$$

$H_n := \{\underline{u} \in H^1(\Omega)^3 : \underline{u} \wedge \underline{n} = 0 \text{ on } \partial\Omega\}$

- X_n is curl and div conforming, hence continuous across interfaces $\Rightarrow H_n = X_n$
- H_n is easy to discretise and implement: Cartesian product of scalar discretisation $S^{1,\underline{p}}(\Omega, \mathcal{T})$ of $H^1(\Omega)$
- Converges to wrong solutions if Ω has reentrant corners:
  - $H_n \neq X_n$
  - $\operatorname{codim}_{X_n} H_n = \infty$
  - H_n closed in X_n ie., sequences in H_n have their limits in H_n.

Weighted Regularization

Find the frequencies $\omega > 0$ such that $\exists \underline{E} \in H_n \setminus \{0\}$ with

$$\int_\Omega \operatorname{curl} \underline{E} \cdot \operatorname{curl} \underline{F} + s\langle \underline{E}, \underline{F} \rangle_Y = \omega^2 \int_\Omega \underline{E} \cdot \underline{F} \qquad \forall \underline{F} \in H_n$$

$$\langle \underline{E}, \underline{F} \rangle_Y = \int_\Omega \rho(\underline{x}) \operatorname{div} \underline{E} \operatorname{div} \underline{F}$$

Properly chosen weight $\rho(\underline{x})$ and $s \in \mathbb{R}_+$.

Weighted Regularization

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$$\int_\Omega \operatorname{curl} \underline{E} \cdot \operatorname{curl} \underline{F} + s\langle \underline{E}, \underline{F} \rangle_Y = \omega^2 \int_\Omega \underline{E} \cdot \underline{F} \qquad \forall \underline{F} \in H_n$$

$$\langle \underline{E}, \underline{F} \rangle_Y = \int_\Omega \rho(\underline{x}) \operatorname{div} \underline{E} \operatorname{div} \underline{F}$$

Properly chosen weight $\rho(\underline{x})$ and $s \in \mathbb{R}_+$.

Idea: use spaces

$$X_n[Y] := \{\underline{u} \in H_0(\operatorname{curl}, \Omega) : \operatorname{div} \underline{u} \in Y\} \supset H_n \text{ dense}$$

and the solutions of Maxwell equations $\in X_n[Y]$.

[2] Martin Costabel and Monique Dauge, “Weighted regularization of Maxwell equations in polyhedral domains”, *Numer. Math.* 93 (2), pp. 239–277 (2002).

Chosing the Weight and s

$$s\langle \underline{E}, \underline{F} \rangle_Y = s \int_\Omega \rho(\underline{x}) \operatorname{div} \underline{E} \operatorname{div} \underline{F}$$

2D:
$\rho(\underline{x}) = r^\alpha$ where r is the distance to a reentrant corner and $\alpha \in [0, 2]$ depending on the angle of the reentrant corner: $\alpha \in (2 - 2^\pi/\omega_c, 2]$

s scales the $\langle . , . \rangle_Y$ form. Spurious Eigenvalues get scaled too, real Eigenvalues not. Sensible range: $(0, 30)$. $s = 0$ gives a large kernel since $\operatorname{div} \underline{E} = 0$ is not enforced at all.
$\alpha = 2$ is the limiting case, nice to implement since r^2 is polynomial.

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3D:

$$\rho(\underline{x}) = \operatorname{dist}(\underline{x}, \mathcal{C} \cup \mathcal{E})^\alpha$$

where $\alpha \in [0, 2]$ (depending on angle of edge and cone of corner).

s scales the $\langle .\,, .\rangle_Y$ form. Spurious Eigenvalues get scaled too, real Eigenvalues not. Sensible range: $(0, 30)$. $s = 0$ gives a large kernel since $\operatorname{div} \underline{E} = 0$ is not enforced at all.
$\alpha = 2$ is the limiting case, nice to implement since r^2 is polynomial.

Convergence of Eigenvalues

As before:
Eigenvectors:

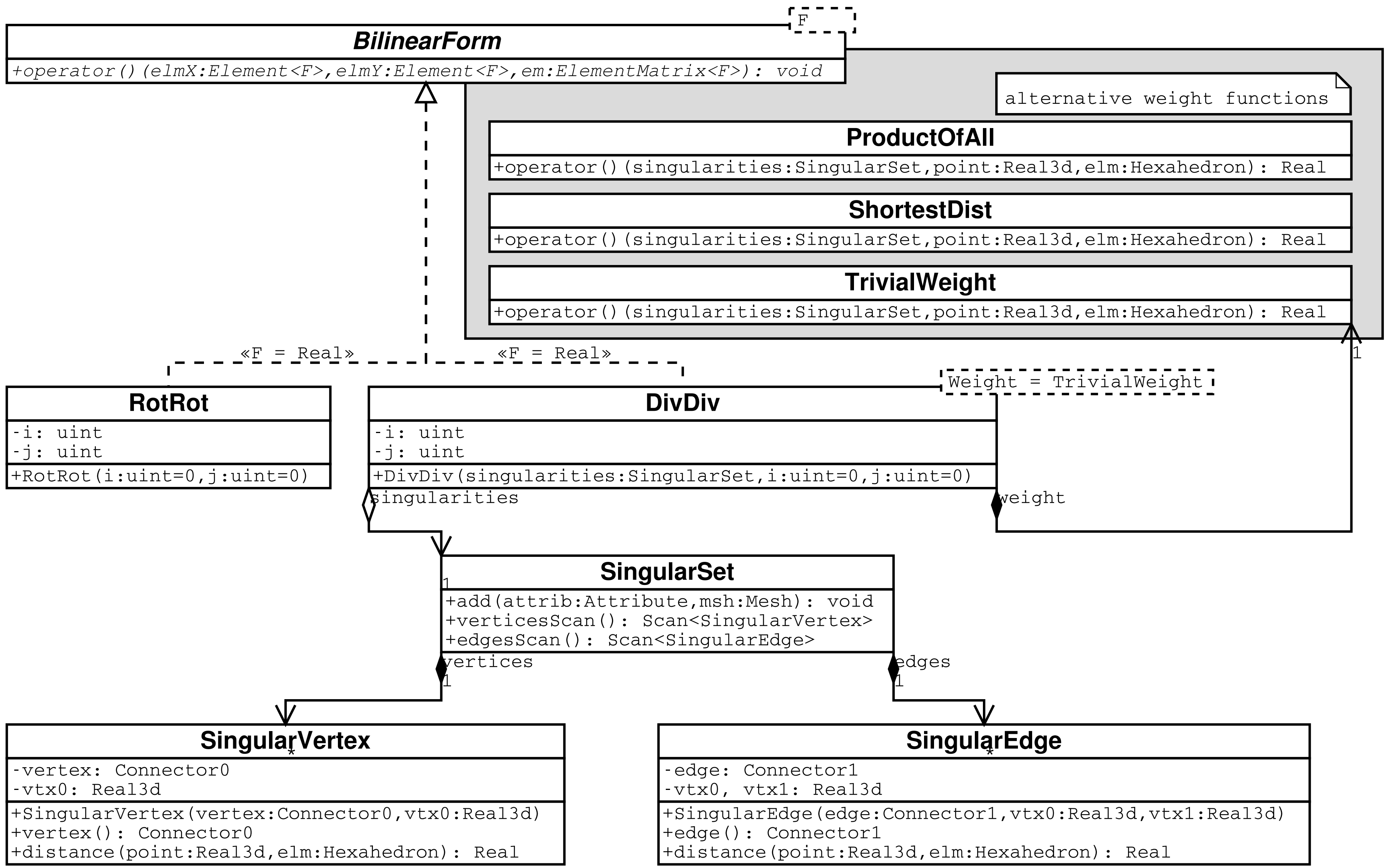
$$\|E_m - E_{m,N}\|_{X_n} \leq C \sup_{F \in W_m} \inf_{F_N \in V_N} \|F - F_N\|_{X_n}$$

Simple Eigenvalues:

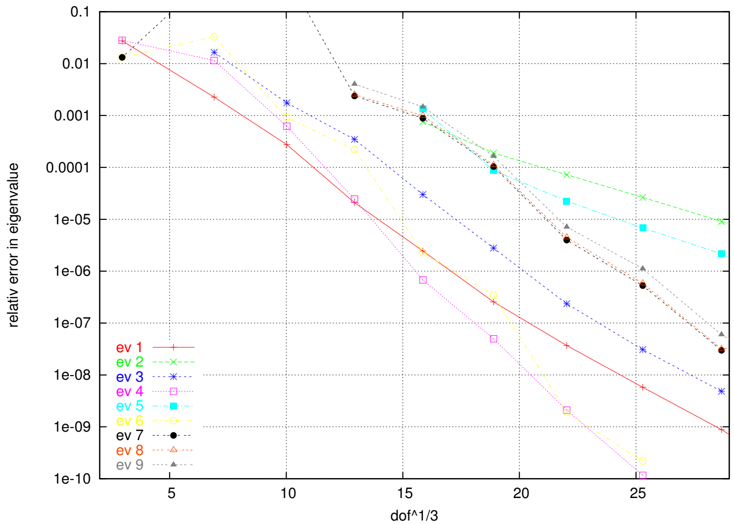
$$|\lambda_m - \lambda_{m,N}| \leq C \sup_{F \in W_m} \inf_{F_N \in V_N} \|F - F_N\|^2_{X_n}$$

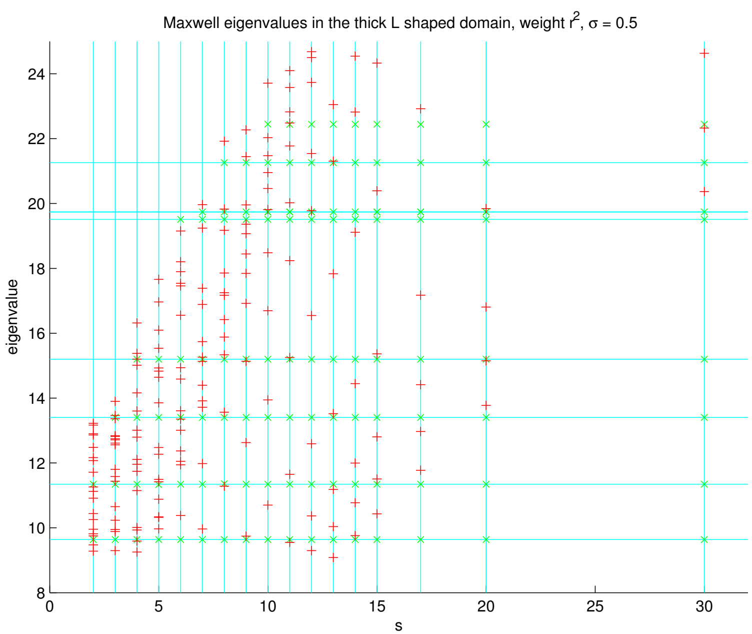
For $\|F - F_N\|_{X_n}$, exponential convergence possible:
$\mathbb{R}^2$: Proof by Costabel, Dauge, Schwab
$\mathbb{R}^3$: experimental evidence, proof in preparation

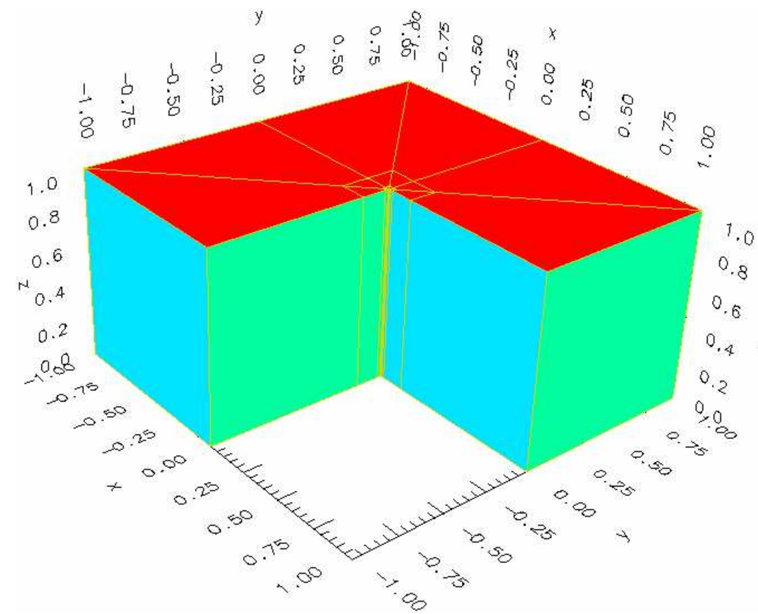
Classes for Maxwell Discretisation



EVP in the Thick L Shaped Domain



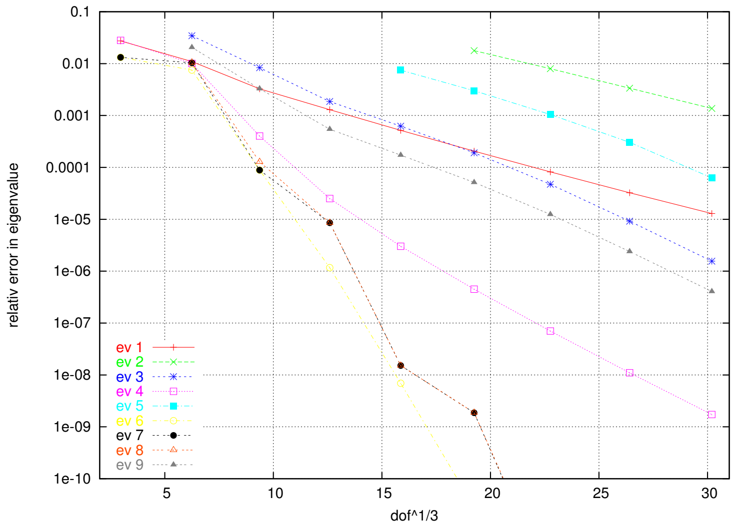


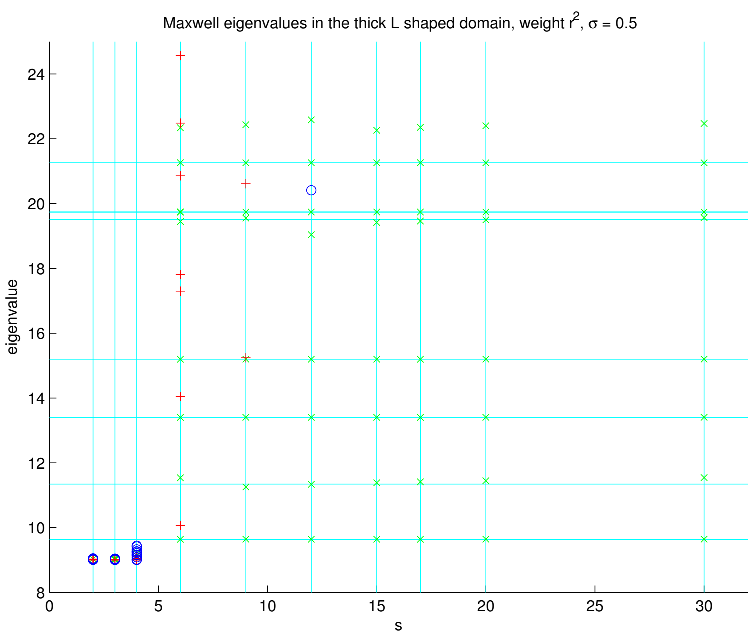


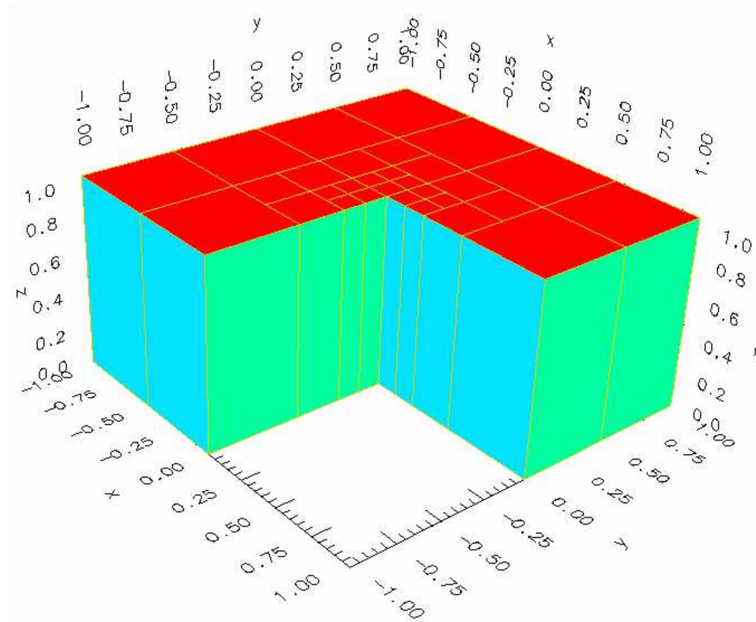
$$\sigma = 0.15$$

$$\alpha = 2$$

EVP in the Thick L Shaped Domain







$$\sigma = 0.5$$

$$\alpha = 2$$

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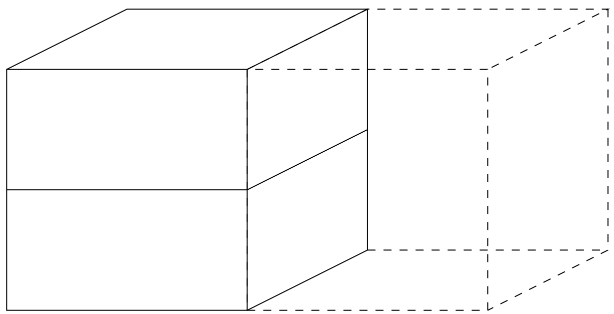
Perspectives

- Maxwell EVP in the Fichera corner
- Maxwell source problems
- A posteriori error estimation, anisotropic regularity estimation
- Improved mesh handling



- Iterative multilevel domain decompositioning solvers:
  Toselli (Zürich), Schöberl (Linz)

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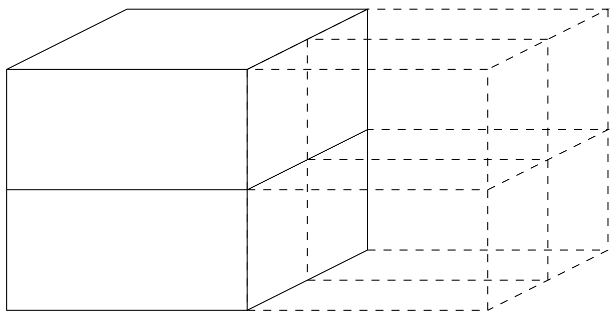

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