Anisotropic h and p refinement for conforming FEM in 3D

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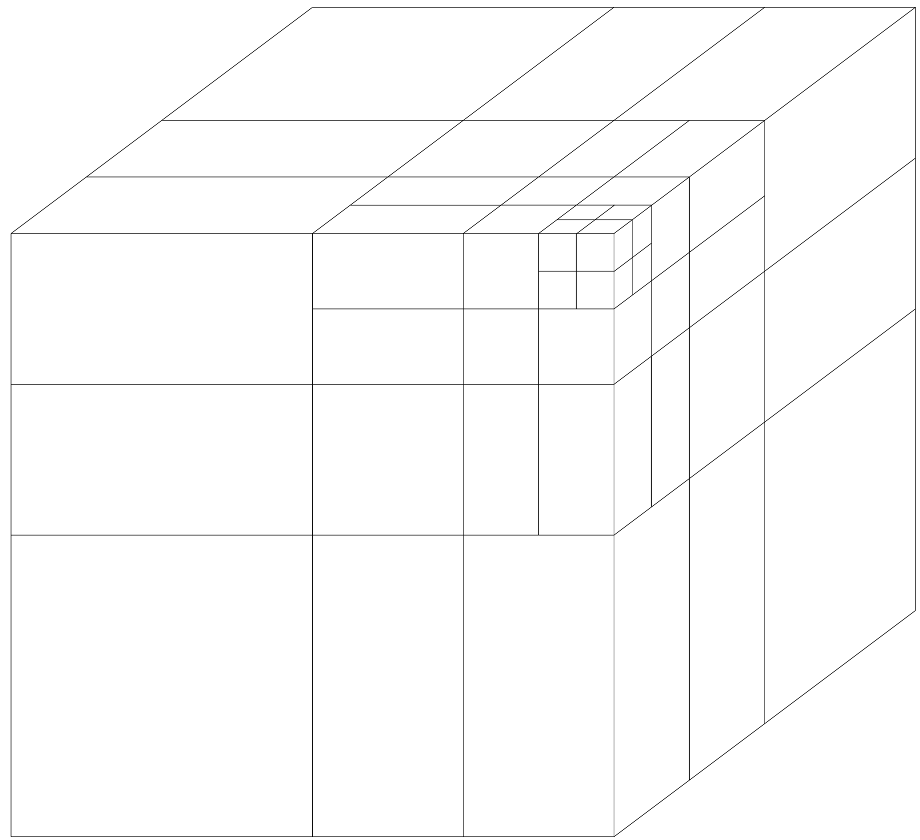
Goal for Meshes



- Hierarchy of hanging nodes
- Anisotropic refinements

Overview

- Introduction
- Anisotropic h refinements
  - S and T matrices
  - Assembly of Supports

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- hp Meshes
- Perspectives

Previous hp Software

- Szabó 1985: PROBE (p only)
- Demkowicz, Oden, Rachowicz et al. 1989: PHLEX, hp90
- Anderson: STRIPE (p only on a-priori generated meshes)
- Flaherty, Shephard: Tetrahedra only (3D anisotropy?)
- Karniadakis, Sherwin: NEKTAR (regular meshes only, tetrahedra, hexahedra, prisms, p only)
- Devloo
- Szabó since 1995: STRESSCHECK (p only)
- Heuveline et al.: HiFlow
- In development: deal.II (Kanschat & Bangerth), ngsolve (Schöberl et al.)

FE Method

- Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ (dimension independent design)
- Find $u \in V$ such that

$$a(u, v) = l(v) \quad \forall v \in V,$$

  V a FE space, $a(.\,,.)$ a bilinear form and $l(.)$ a linear form.
- Standard FE: $V \subset H^1(\Omega)$

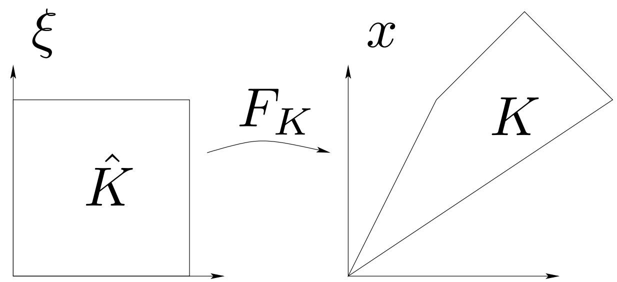
$$\begin{aligned} V &= S^{1,\underline{p}}(\Omega, \mathcal{T}) \\ &= \left\{ u \in H^1(\Omega) : u|_K \circ F_K \in \mathcal{Q}_p \ \forall K \in \mathcal{T} \right\} \end{aligned}$$

  $\Rightarrow u \in V$ is continuous, ie. $\mathcal{C}^0(\bar{\Omega})$.
- Vector valued problems are possible

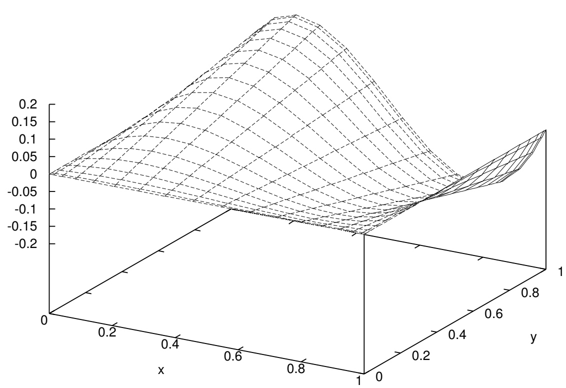
FE Space: Generalities

- Basis $\{\Phi_i\}_{i=1}^{N}$ constructed from element shape functions ϕ_j^K on elements $K \in \mathcal{T}$.
- Reference element shape functions: N_j, element map: $F_K : \hat{K} \to K$

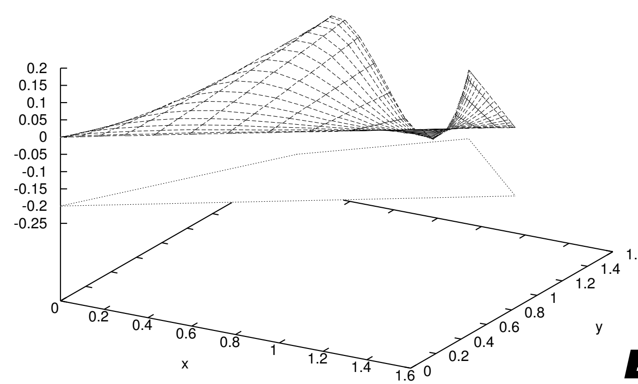
$$\Rightarrow \phi_j^K \circ F_K = N_j .$$



N_j

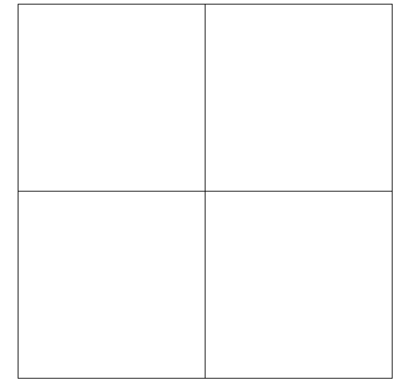


Φ_j



FE Meshes

Local refinements as mean to improve approximation of exact solution by FE solution



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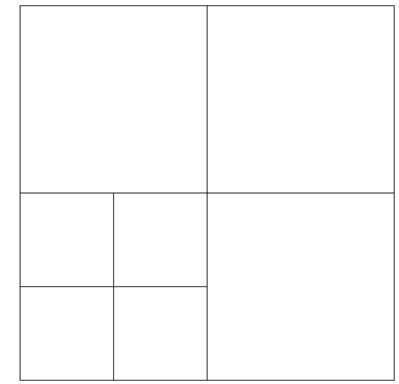

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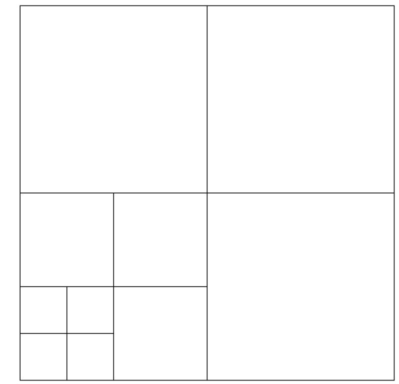

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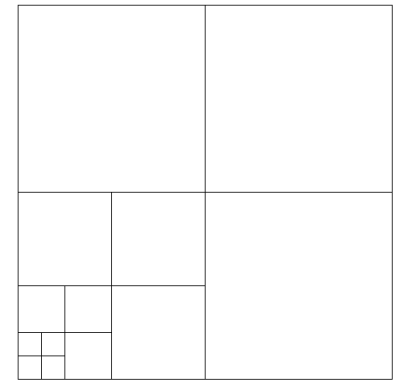

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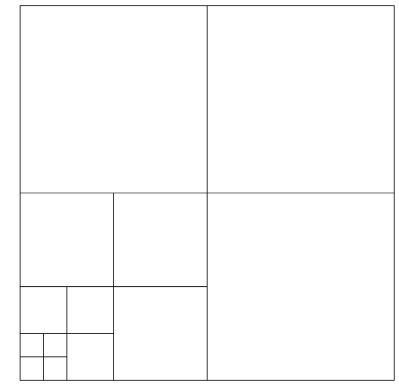
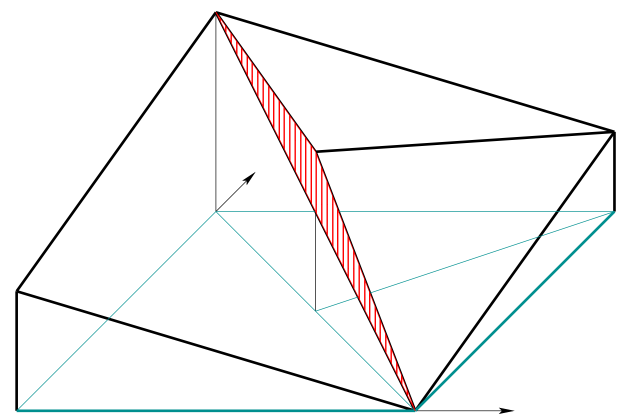
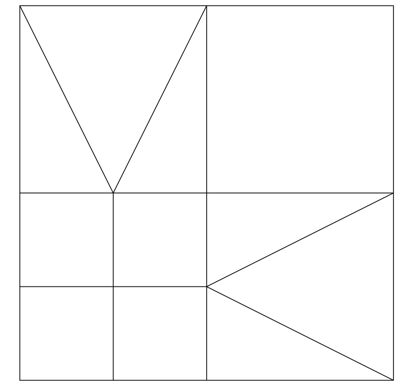


But standard FE forbids locally refined grids: discontinuities are possible.



Mortar vs. Enforcing Continuity



- Topolocigal closure

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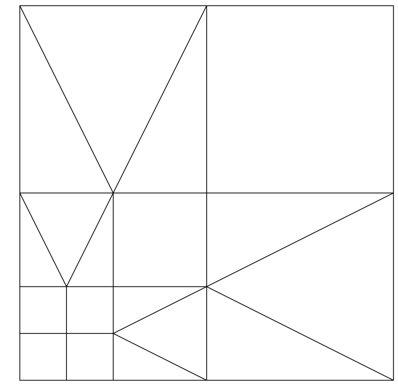

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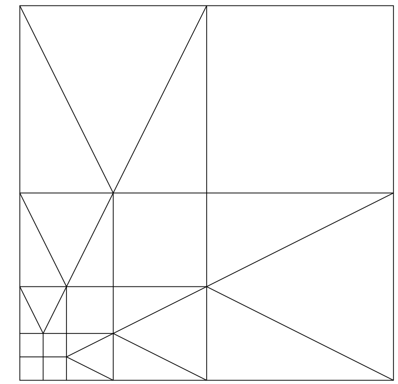

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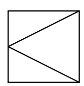
  Drawbacks: more elements, more element types, what about refining a  ?

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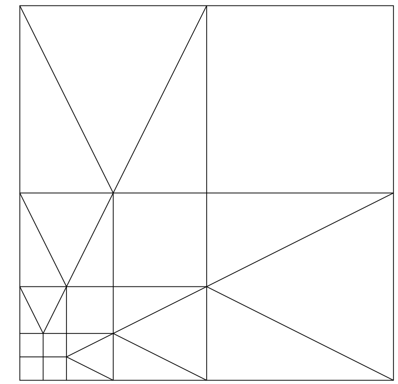

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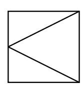
  Drawbacks: more elements, more element types, what about refining a  ?
- Our philosophy: hexahedral meshes only (tensorized interpolants, spectral quadrature techniques)
- Our solution: Treating the constraints induced by the hanging nodes
  Why conforming? $a(u, v) = a(v, u)$ and $a(u, u) \geq \alpha \|u\|_V^2 \Rightarrow \boldsymbol{A}$ SPD, pccg . . .

Our Software: Concepts

- Started by Christian Lage during his Ph.D. studies (1995).
- Used and improved by Frauenfelder, Matache, Schmidlin, Schmidt and several students.
- Concept Oriented Design using mathematical principles [1].
- Currently two parts: hp-FEM, BEM (wavelet and multipole methods).
- C++

[1] P. F. and Ch. Lage, "Concepts—An Object Oriented Software Package for Partial Differential Equations", *Mathematical Modelling and Numerical Analysis* 36 (5), pp. 937–951 (2002).

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T Matrix

Definition 1 (T Matrix). *Element shape functions* $\{\phi_j^K\}_{j=1}^{m_K}$ *on element* K, *global basis functions* $\{\Phi_i\}_{i=1}^{N}$.
The T matrix $\boldsymbol{T}_K \in \mathbb{R}^{m_K \times N}$ *of element* K *is implicitly defined by*

$$\Phi_i|_K = \sum_{j=1}^{m_K} [\boldsymbol{T}_K]_{ji}\, \phi_j^K$$

as vectors:

$$\underline{\Phi}|_K = \boldsymbol{T}_K^\top \underline{\phi}^K.$$

Assembly using T Matrices

Assembling:

$$\underline{l} = l(\underline{\Phi}) = l\Big(\sum_{\tilde{K}} \boldsymbol{T}_{\tilde{K}}^{\top} \underline{\phi}^{\tilde{K}}\Big) = \sum_{\tilde{K}} \boldsymbol{T}_{\tilde{K}}^{\top} l(\underline{\phi}^{\tilde{K}}) = \sum_{\tilde{K}} \boldsymbol{T}_{\tilde{K}}^{\top} \underline{l}_{\tilde{K}}$$

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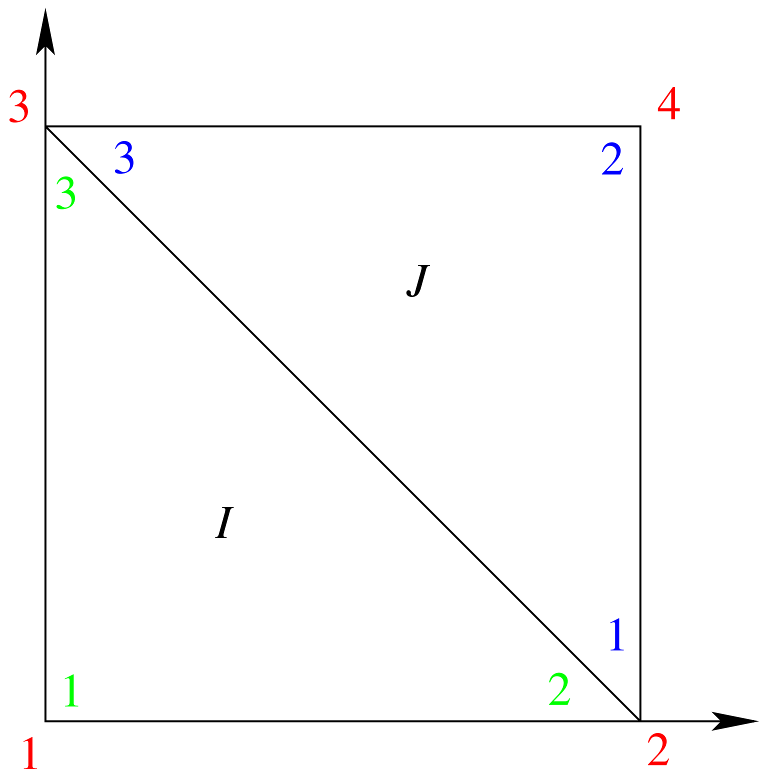
$$\boldsymbol{A} = a(\underline{\Phi}, \underline{\Phi}) = \sum_{K,\tilde{K}} \boldsymbol{T}_{\tilde{K}}^{\top} a(\underline{\phi}^{K}, \underline{\phi}^{\tilde{K}}) \boldsymbol{T}_{K} = \sum_{K,\tilde{K}} \boldsymbol{T}_{\tilde{K}}^{\top} \boldsymbol{A}_{\tilde{K}K} \boldsymbol{T}_{K}$$

Note: $\boldsymbol{A}_{\tilde{K}K} = 0$ in standard FEM for $\tilde{K} \neq K$.

Example: Regular Mesh

Two elements with three local shape functions each and four global basis functions.



$$T_I = \begin{pmatrix} & 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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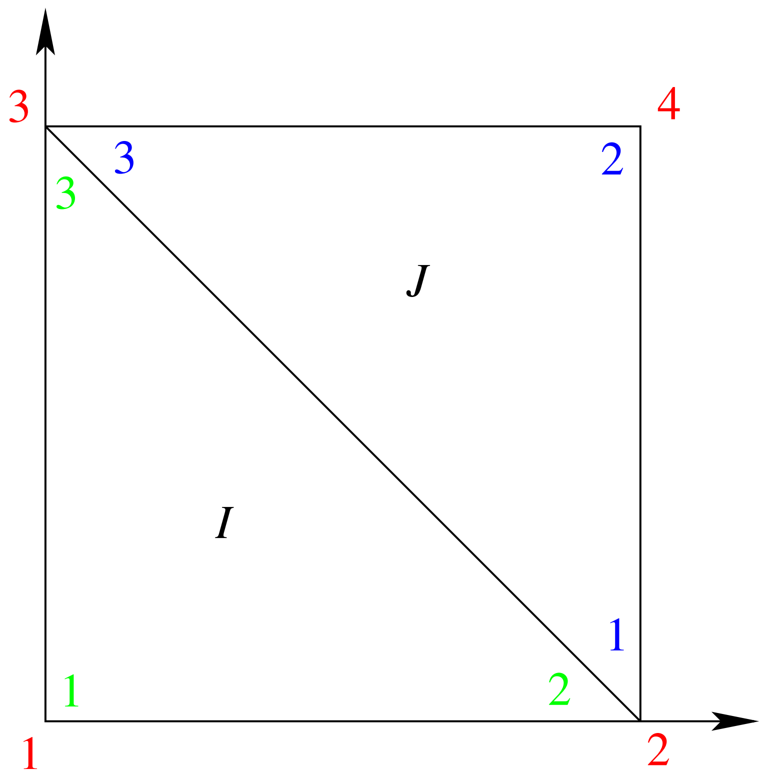

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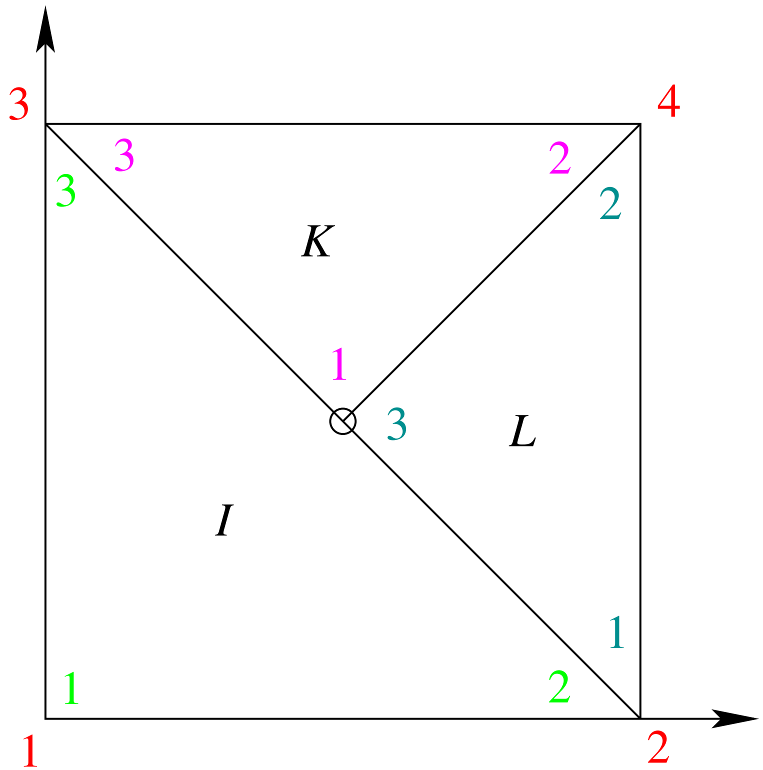
$$

$$
\boldsymbol{T}_J = \begin{pmatrix} & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 \end{pmatrix}
$$

Example: Irregular Mesh

Three elements with three local shape functions each and four global basis functions. The hanging node is marked with $\circ$.



$$T_L = \begin{pmatrix} & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

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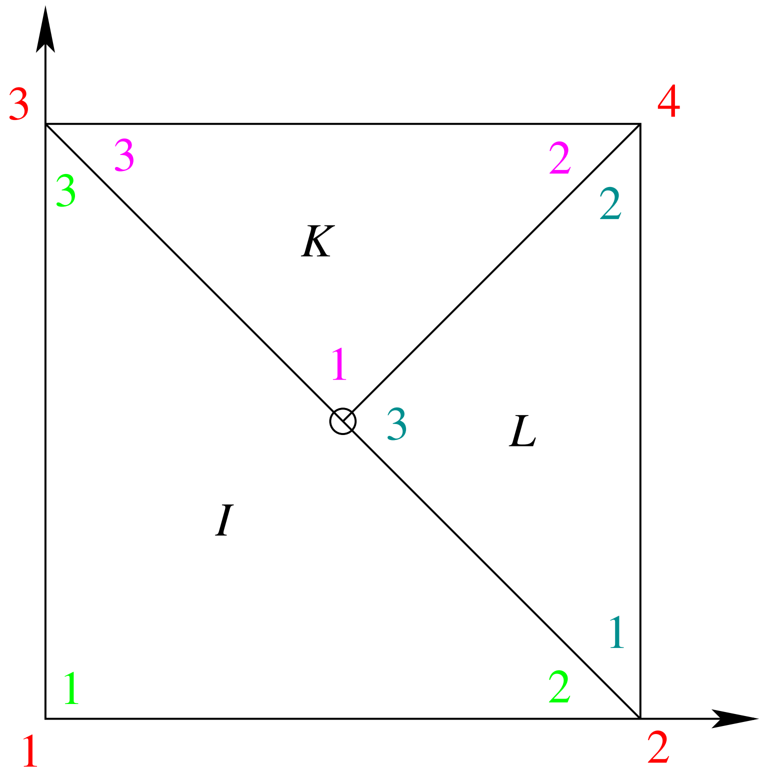

$$\boldsymbol{T}_L = \begin{pmatrix} & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

$$\boldsymbol{T}_K = \begin{pmatrix} & 1 & 2 & 3 & 4 \\ 1 & 0 & 1/2 & 1/2 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$\Rightarrow$ continuous basis functions.

Generation of T Matrices

- **Regular Mesh:** Counting and assigning indices with respect to topological entities such as vertices, edges and faces. Explained in detail later.

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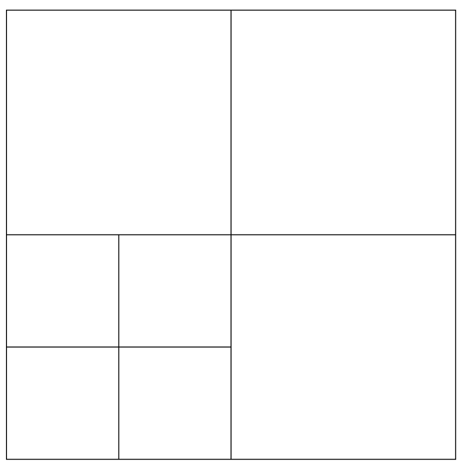
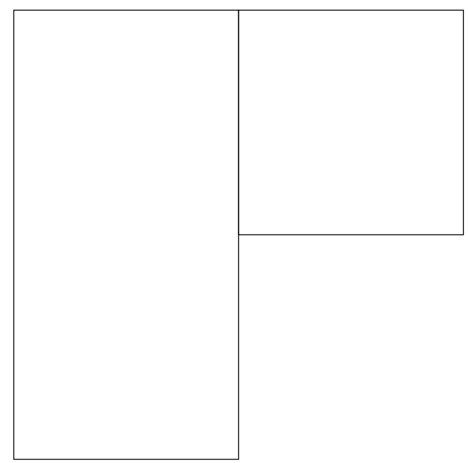
- **Irregular Mesh:** Irregularity due to a refinement of an initially regular mesh.

OK: , not OK: 

Explanation follows.

T Matrices for Irregular Meshes

Irregularity due to a refinement of an initially regular mesh.

Mesh	$\mathcal{M}$	refine	$\mathcal{M}'$
Basis fcts.	$B = B_{\text{repl}} \cup B_{\text{keep}}$	$\longrightarrow$	$B' = B_{\text{ins}} \cup B_{\text{keep}}$

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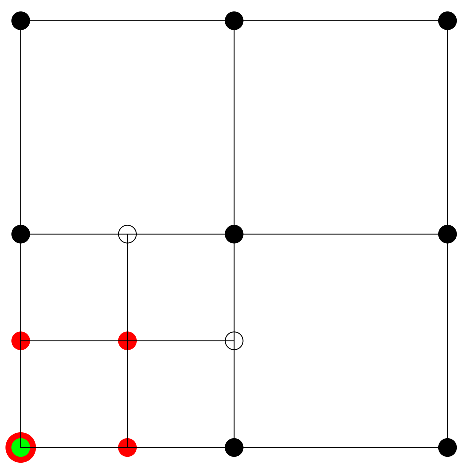

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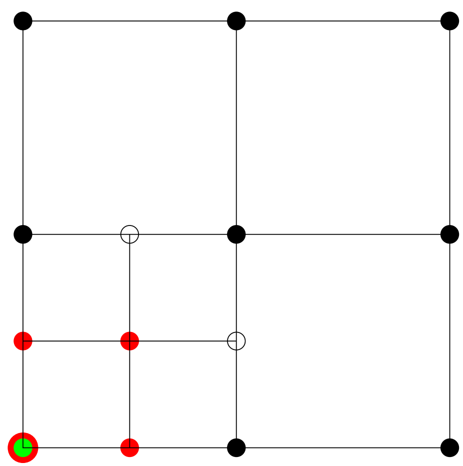

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Every element of B has a column in the T matrix. Generation is

- easy for B_{ins} (like regular mesh),
- simple for B_{keep}: modify column from $\mathcal{M}$ by S matrix.

S Matrix

Definition 2 (S Matrix). *Let $K' \subset K$ be the result of a refinement of element K. The S matrix $\boldsymbol{S}_{K'K} \in \mathbb{R}^{m_{K'} \times m_K}$ is defined by*

$$\phi_j^K \Big|_{K'} = \sum_{l=1}^{m_{K'}} [\boldsymbol{S}_{K'K}]_{lj} \, \phi_l^{K'}$$

as vectors:

$$\underline{\phi}^K \Big|_{K'} = \boldsymbol{S}_{K'K}^\top \underline{\phi}^{K'}$$

$\phi_j^K \big|_{K'}$ *is represented as a linear combination of the shape functions* $\left\{\phi_l^{K'}\right\}_{l=1}^{m_{K'}}$ *of* K'.

Application of S Matrix

Proposition 1. *Let* $K' \subset K$ *be the result of a refinement of an element* K*. Then, the T matrix of* K' *can be computed as*

$$\boldsymbol{T}_{K'} = \boldsymbol{S}_{K'K}\boldsymbol{T}_{K}^{\text{keep}} + \boldsymbol{T}_{K'}^{\text{ins}}$$

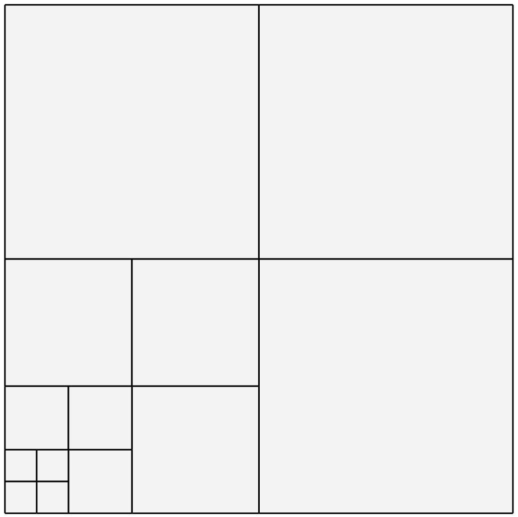
where $\boldsymbol{T}_{K}^{\text{keep}}$ *denotes the T matrix of element* K *(with columns not related to functions in* B_{keep} *set to zero) and* $\boldsymbol{T}_{K'}^{\text{ins}}$ *the T matrix for functions in* B_{ins} *with respect to* K'*.*

Proposition 2. *Let* $\hat{K}' \subset \hat{K}$ *be the result of a refinement of the reference element* $\hat{K}$ *with* $H : \hat{K} \to \hat{K}'$ *the subdivision map. The element maps are*

$$F_K : \hat{K} \to K \text{ and } F_{K'} : \hat{K} \to K'$$

and $F_{K'} \circ H^{-1} = F_K$ *holds. Then,* $\boldsymbol{S}_{\hat{K}'\hat{K}} = \boldsymbol{S}_{K'K}$*.*

Meshes



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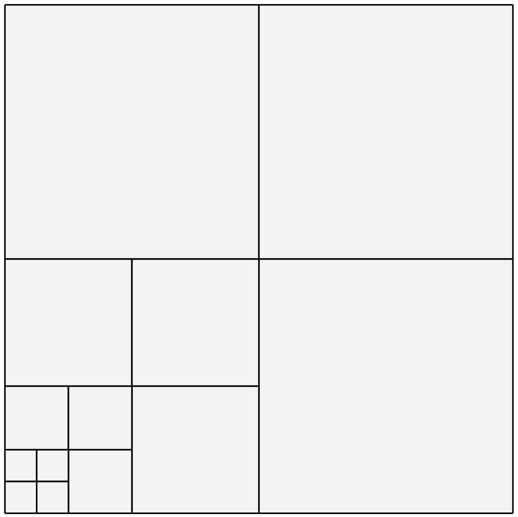

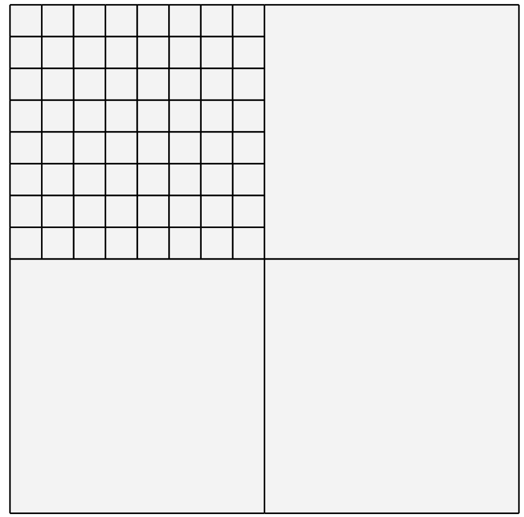

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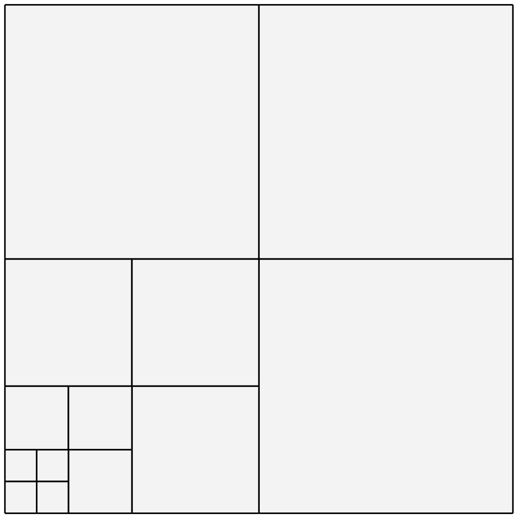

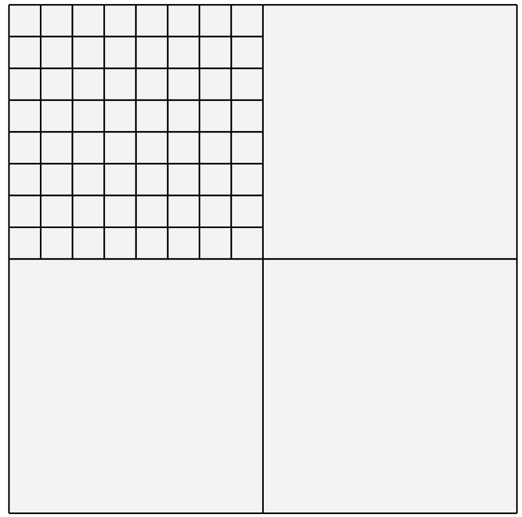

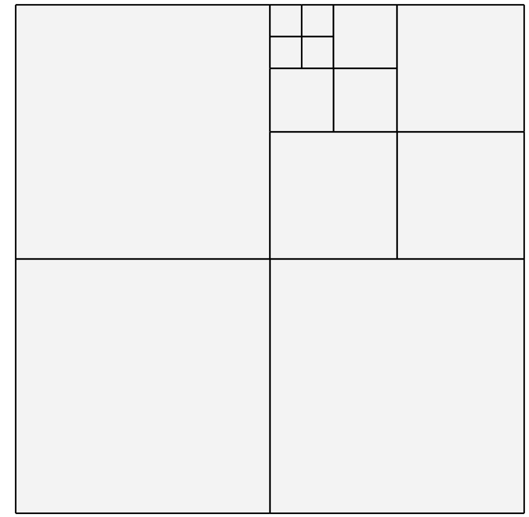

S Matrix in Dimension $d = 1$

Subdividing $\hat{J} = (0,1)$ in $\hat{J}' = (0, 1/2)$ and $\hat{J}^\star = (1/2, 1)$ with the reference element shape functions

$$N_j(\xi) = \begin{cases} 1-\xi & j = 1 \\ \xi & j = 2 \\ \xi(1-\xi)P_{j-3}^{1,1}(2\xi - 1) & j = 3,\dots,J \end{cases}$$

yields (solving a linear system) for $J = 4$:

$$\boldsymbol{S}_{\hat{J}',\hat{J}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 1/4 & 0 \\ 0 & 0 & 1/4 & -3/4 \\ 0 & 0 & 0 & 1/8 \end{pmatrix} \text{ and } \boldsymbol{S}_{\hat{J}^\star,\hat{J}} = \begin{pmatrix} 1/2 & 1/2 & 1/4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 1/8 \end{pmatrix}.$$

Hierarchic shape functions $\Rightarrow$ hierarchic S matrices.

S Matrices: Tensor Product in 2D

- $d > 1$ with hexahedral meshes $\Rightarrow$ S matrices are built from tensor products of 1D S matrices.

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- In 2D: $N_{i,j} = N_i \otimes N_j$, the four bilinear shape functions are:

$$N_{1,2}(\underline{\xi}) = N_1(\xi_1) \cdot N_2(\xi_2) \qquad N_{2,2}(\underline{\xi}) = N_2(\xi_1) \cdot N_2(\xi_2)$$

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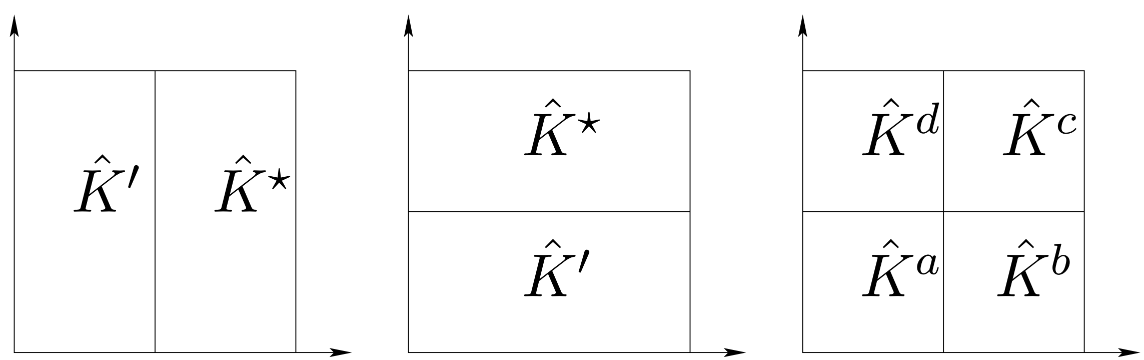
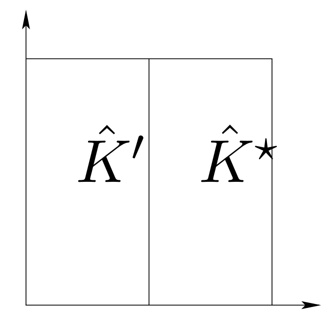
- Consider the subdivisions:



S Matrices: Tensor Product in 2D II

Subdivision map of left variant: $H : \hat{K} \to \hat{K}', \underline{\xi} \mapsto \begin{pmatrix} \xi_1/2 \\ \xi_2 \end{pmatrix}$. S matrix $\boldsymbol{S}_{\hat{K}'\hat{K}}$ is defined by:

$$N_{i,j}|_{\hat{K}'} = \sum_{k,l} \left[\boldsymbol{S}_{\hat{K}'\hat{K}}\right]_{(k,l),(i,j)} N_{k,l} \circ H^{-1}.$$



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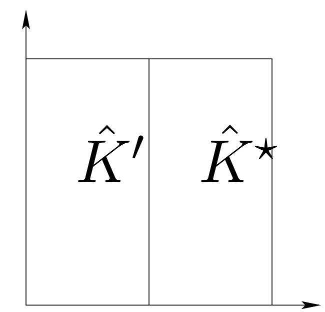

Tensor product shape functions:

$$(N_i \otimes N_j)|_{\hat{K}'} = \sum_{k,l} \left[\boldsymbol{S}_{\hat{K}'\hat{K}}\right]_{(k,l),(i,j)} (N_k \otimes N_l) \circ H^{-1}. \quad (1)$$

S Matrices: Tensor Product in 2D III

S matrices for 1D reference element shape fcts. used in (1):

$$N_i|_{\hat{J}'} = \sum_m \left[\boldsymbol{S}_{\hat{J}',\hat{J}}\right]_{mi} N_m \circ G^{-1}$$ for the ξ_1 part and

$$N_j = \sum_n [\boldsymbol{E}]_{nj} N_n$$ for the ξ_2 part,

where $G : \xi \mapsto \xi/2$.

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where $G : \xi \mapsto \xi/2$. Plugging into the left hand side of (1) yields:

$$\begin{aligned}(N_i \otimes N_j)|_{\hat{K}'} = N_i|_{\hat{j}'} \otimes N_j &= \sum_{m,n} \left(\left[\boldsymbol{S}_{\hat{j}',\hat{j}}\right]_{mi} N_m \circ G^{-1}\right) \otimes \Big([\boldsymbol{E}]_{nj} N_n\Big) \\ &= \sum_{m,n} \left[\boldsymbol{S}_{\hat{j}',\hat{j}}\right]_{mi} \cdot [\boldsymbol{E}]_{nj} N_m \circ G^{-1} \otimes N_n .\end{aligned}$$

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S Matrices: Tensor Product in 2D IV

Comparing with the right hand side of (1):

$$\sum_{m,n} \left[\boldsymbol{S}_{\hat{J}',\hat{J}}\right]_{mi} \cdot [\boldsymbol{E}]_{nj}\, N_m \circ G^{-1} \otimes N_n$$

$$= \sum_{k,l} \left[\boldsymbol{S}_{\hat{K}'\hat{K}}\right]_{(k,l),(i,j)} N_k \circ G^{-1} \otimes N_l.$$

S Matrices: Tensor Product in 2D IV

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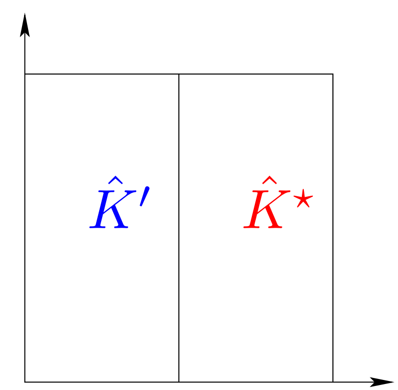
$$\sum_{m,n} \left[\boldsymbol{S}_{\hat{J}',\hat{J}}\right]_{mi} \cdot [\boldsymbol{E}]_{nj}\, N_m \circ G^{-1} \otimes N_n$$

$$= \sum_{k,l} \left[\boldsymbol{S}_{\hat{K}'\hat{K}}\right]_{(k,l),(i,j)} N_k \circ G^{-1} \otimes N_l .$$

Therefore for the vertical subdivision:

$\boldsymbol{S}_{\hat{K}'\hat{K}} = \boldsymbol{S}_{\hat{J}',\hat{J}} \otimes \boldsymbol{E}$ for the left quad $\hat{K}'$,

$\boldsymbol{S}_{\hat{K}^\star\hat{K}} = \boldsymbol{S}_{\hat{J}^\star,\hat{J}} \otimes \boldsymbol{E}$ for the right quad $\hat{K}^\star$.

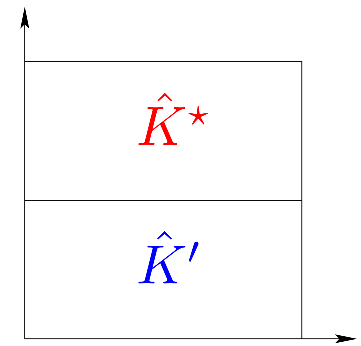


S Matrices: Tensor Product in 2D V

Horizontal subdivision:

$$\boldsymbol{S}_{\hat{K}'\hat{K}} = \boldsymbol{E} \otimes \boldsymbol{S}_{\hat{J}'\hat{J}} \qquad \text{for the bottom quad } \hat{K}',$$

$$\boldsymbol{S}_{\hat{K}^\star\hat{K}} = \boldsymbol{E} \otimes \boldsymbol{S}_{\hat{J}^\star\hat{J}} \qquad \text{for the top quad } \hat{K}^\star.$$

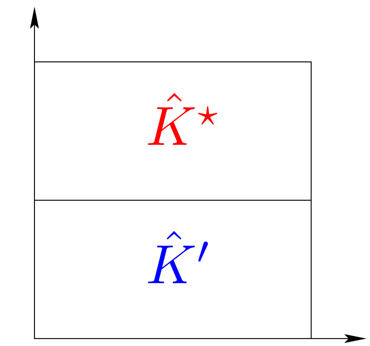


S Matrices: Tensor Product in 2D V

Horizontal subdivision:

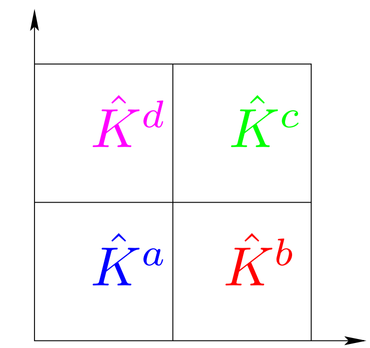
$$\boldsymbol{S}_{\hat{K}'\hat{K}} = \boldsymbol{E} \otimes \boldsymbol{S}_{\hat{J}'\hat{J}}$$ for the bottom quad $\hat{K}'$,

$$\boldsymbol{S}_{\hat{K}^\star\hat{K}} = \boldsymbol{E} \otimes \boldsymbol{S}_{\hat{J}^\star\hat{J}}$$ for the top quad $\hat{K}^\star$.



Subdivision into four quads:

- subdivide $\hat{K}$ horizontally into two children
- subdivide upper and lower child vertically into $\hat{K}^d$ and $\hat{K}^c$ and $\hat{K}^a$ and $\hat{K}^b$ resp.



$$\boldsymbol{S}_{\hat{K}^d\hat{K}} = \left(\boldsymbol{S}_{\hat{J}'\hat{J}} \otimes \boldsymbol{E}\right) \cdot \left(\boldsymbol{E} \otimes \boldsymbol{S}_{\hat{J}^\star\hat{J}}\right) \qquad \boldsymbol{S}_{\hat{K}^c\hat{K}} = \left(\boldsymbol{S}_{\hat{J}^\star\hat{J}} \otimes \boldsymbol{E}\right) \cdot \left(\boldsymbol{E} \otimes \boldsymbol{S}_{\hat{J}^\star\hat{J}}\right)$$

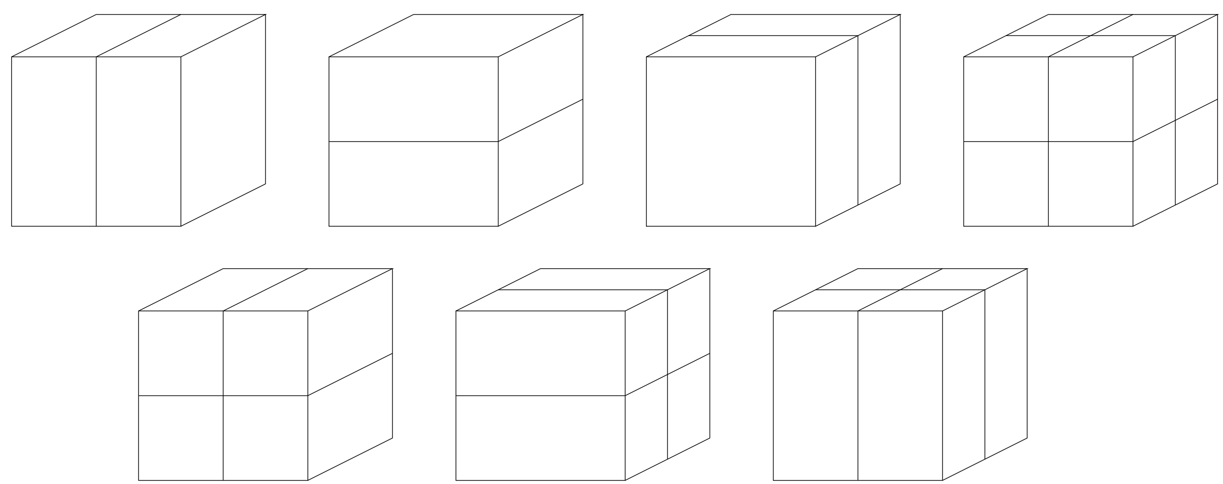
$$\boldsymbol{S}_{\hat{K}^a\hat{K}} = \left(\boldsymbol{S}_{\hat{J}'\hat{J}} \otimes \boldsymbol{E}\right) \cdot \left(\boldsymbol{E} \otimes \boldsymbol{S}_{\hat{J}'\hat{J}}\right) \qquad \boldsymbol{S}_{\hat{K}^b\hat{K}} = \left(\boldsymbol{S}_{\hat{J}^\star\hat{J}} \otimes \boldsymbol{E}\right) \cdot \left(\boldsymbol{E} \otimes \boldsymbol{S}_{\hat{J}'\hat{J}}\right)$$

S Matrices: Tensor-Product in 3D

Same idea as in 2D, just of this form:

$$S_{\hat{K}'\hat{K}} = \prod (A \otimes B \otimes C)$$

in each of the factors, one of A, B or C is an 1D S matrix.
Depending on the factors, 7 subdivisions are possible:



Concepts: allow arbitrary number and combination of these 7 subdivisions

in 3D.

Overview

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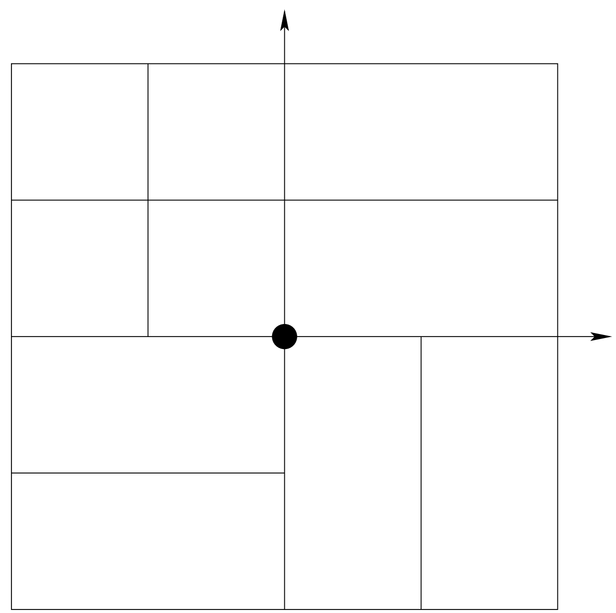
Anisotropic and Conforming

Main point: find the cells (either coarse or fine) which belong to the support of a certain basis function.



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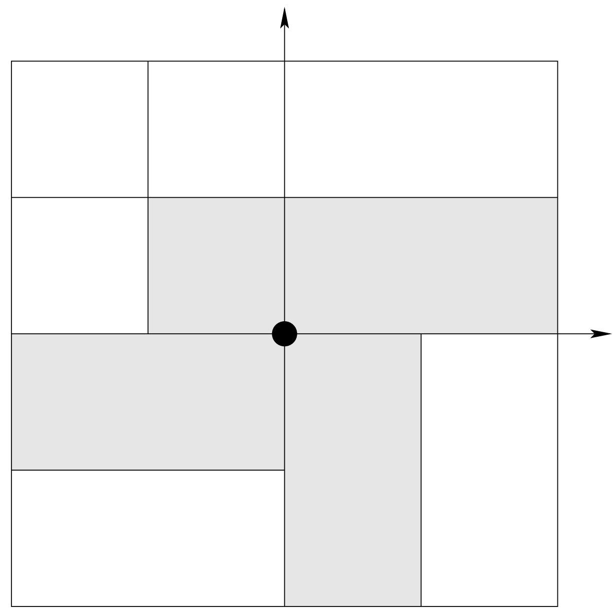

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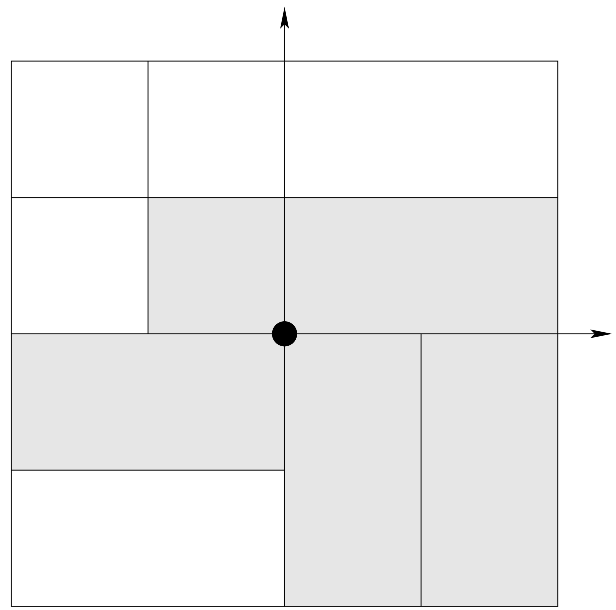

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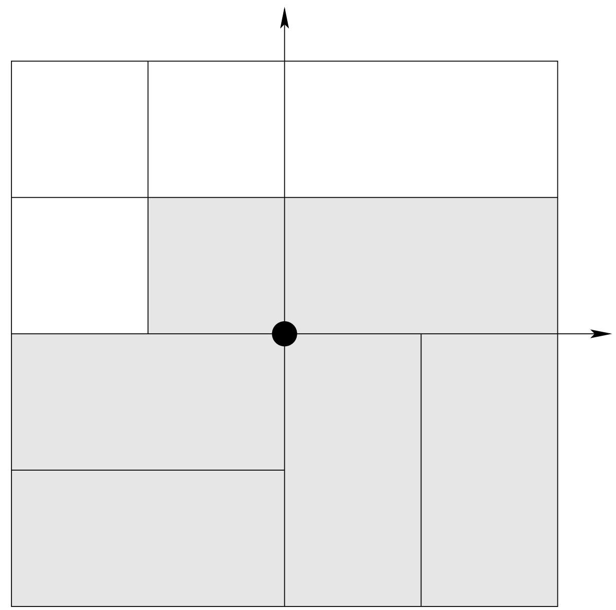

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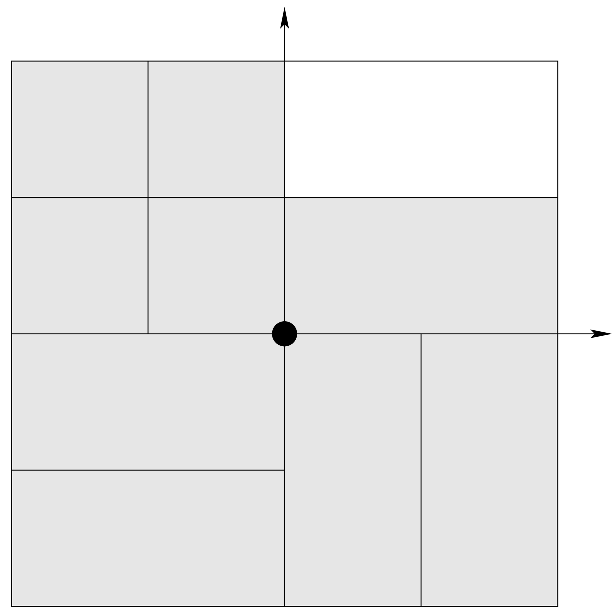

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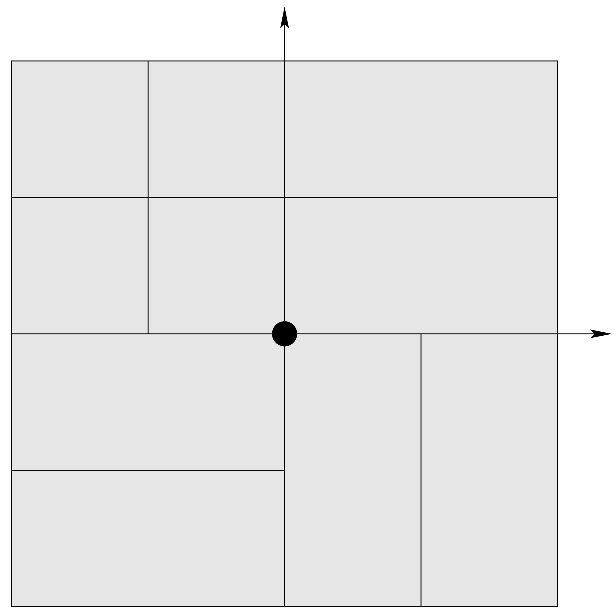
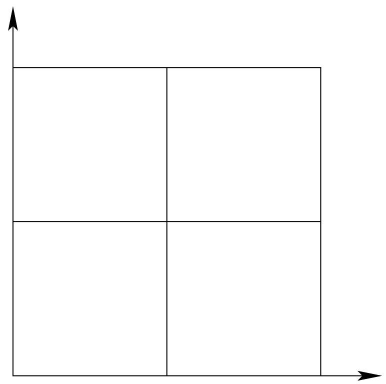


Anisotropic and Conforming II



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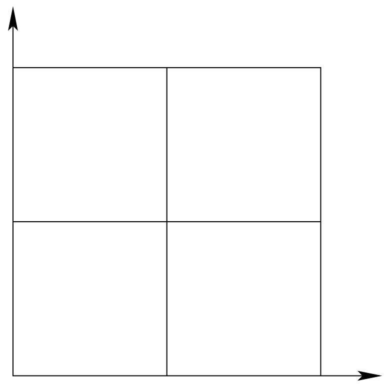

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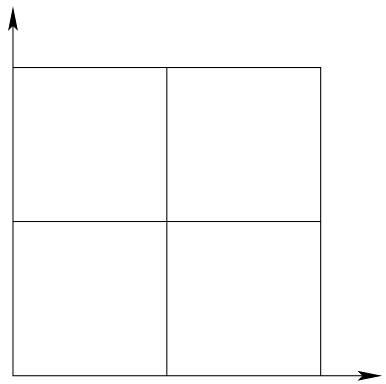

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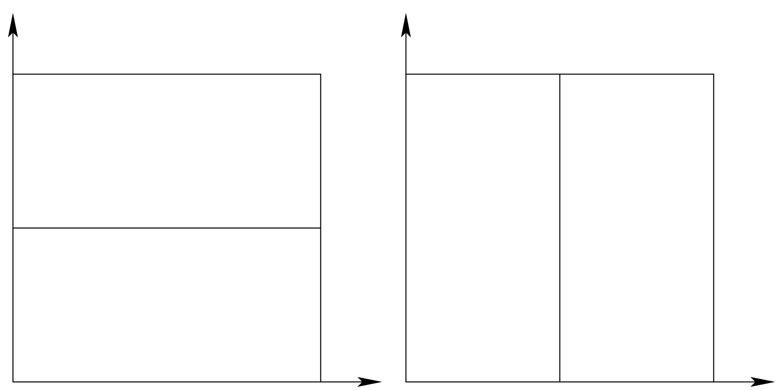

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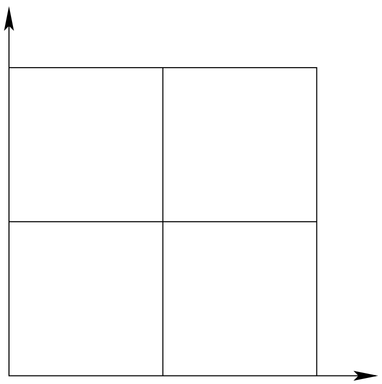
Anisotropic and Conforming II



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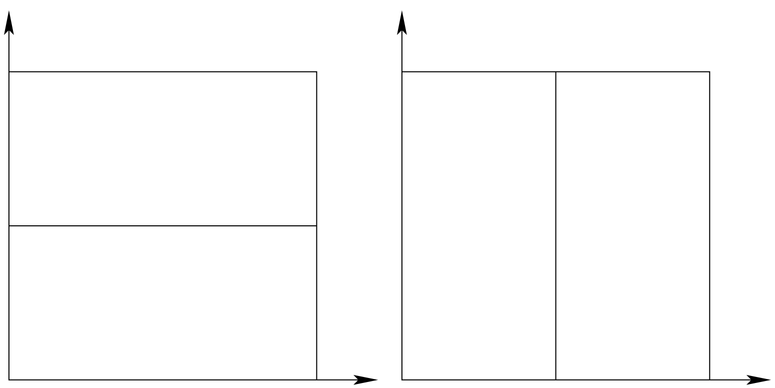

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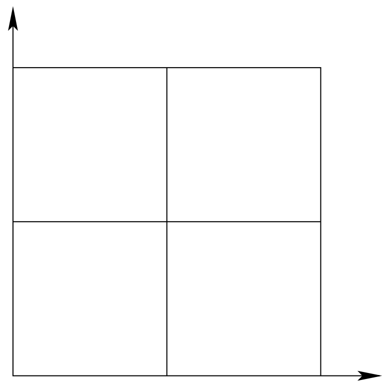

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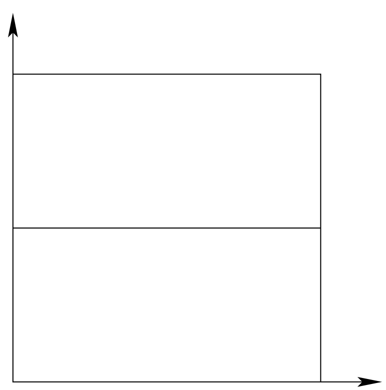

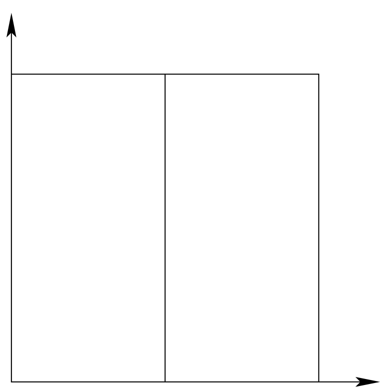

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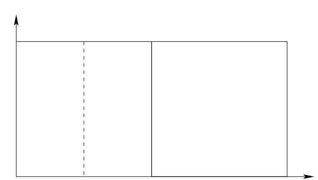
-  should be seen as conforming

Condition for Continuity

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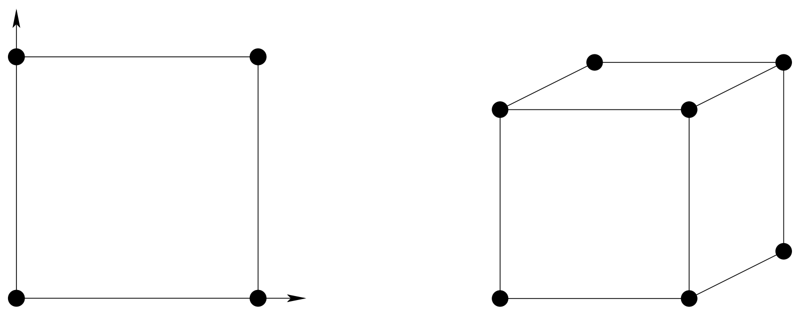

$\Rightarrow$ matching edges which coincide in a vertex is sufficient

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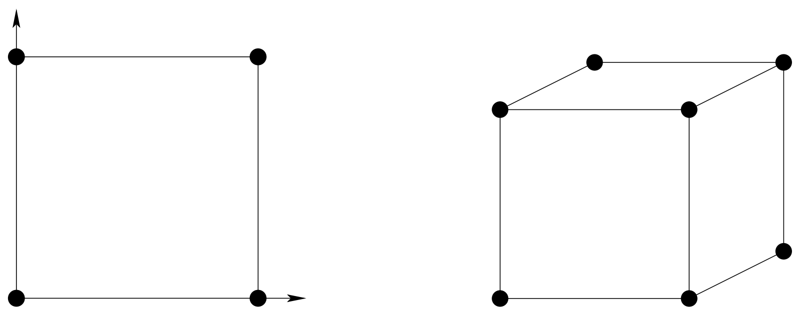


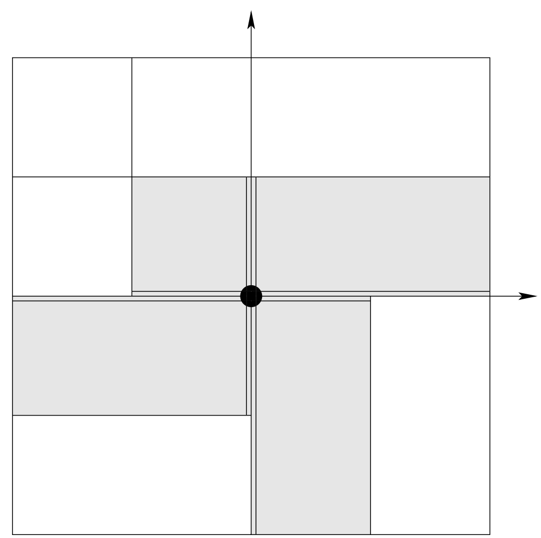
$\Rightarrow$ matching edges which coincide in a vertex is sufficient

- Unisolvent set for $\mathcal{Q}_p$ are
  - additional $p-1$ points on every edge
  - additional $(p-1)^2$ points on every face
  - additional $(p-1)^3$ points in the interior

$\Rightarrow$ matching faces which coincide in an edge is sufficient for continuous edge modes

Algorithm for Continuity (Vertex)

- In every cell of the finest mesh, register all edges and cells in their vertices.



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- For every vertex, while something changed in the last loop:
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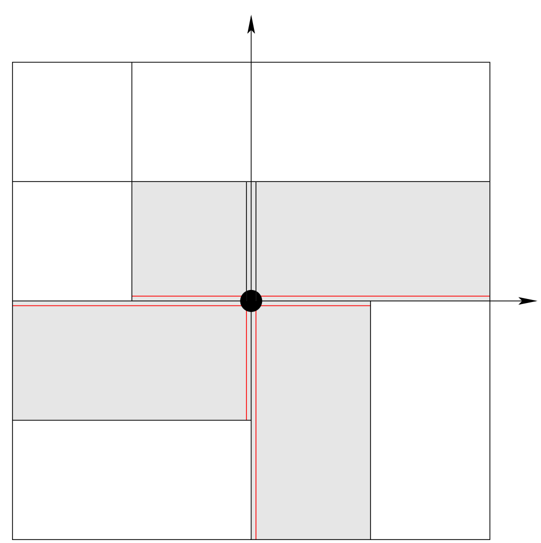

Algorithm for Continuity (Vertex)

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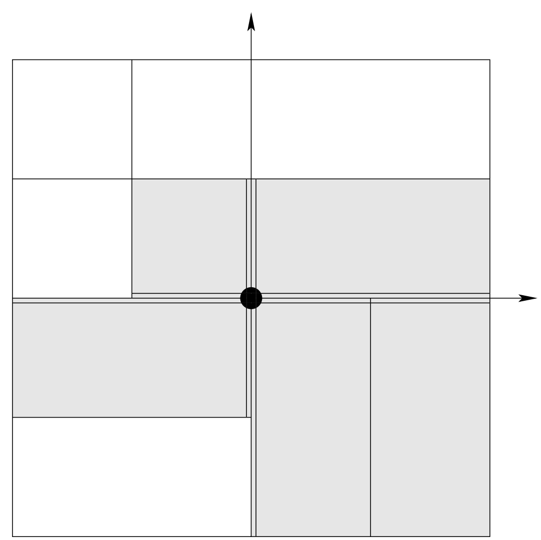
- For every vertex, while something changed in the last loop:
  - Check if some of the edges of the vertex have a relationship (ancestor / descendant).
  - If two edges are related, exchange the smaller cell in the list of the vertex by the cell matching the larger cell.
  - Delete the list of edges and rebuild it from the list of cells.



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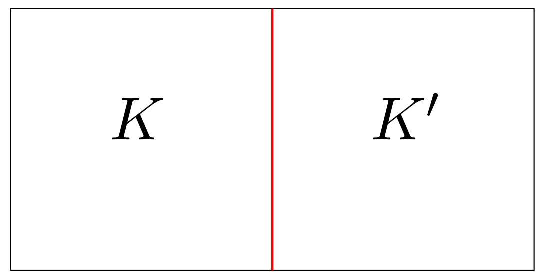
Anisotropic p

Why anisotropic p? Necessary for thin plates, shells, films.

- Every edge has p, every face has $\underline{p} = (p_0, p_1)$, every cell has $\underline{p} = (p_0, p_1, p_2)$. They can differ in a cell!
- The minimum rule for edges and faces is enforced.

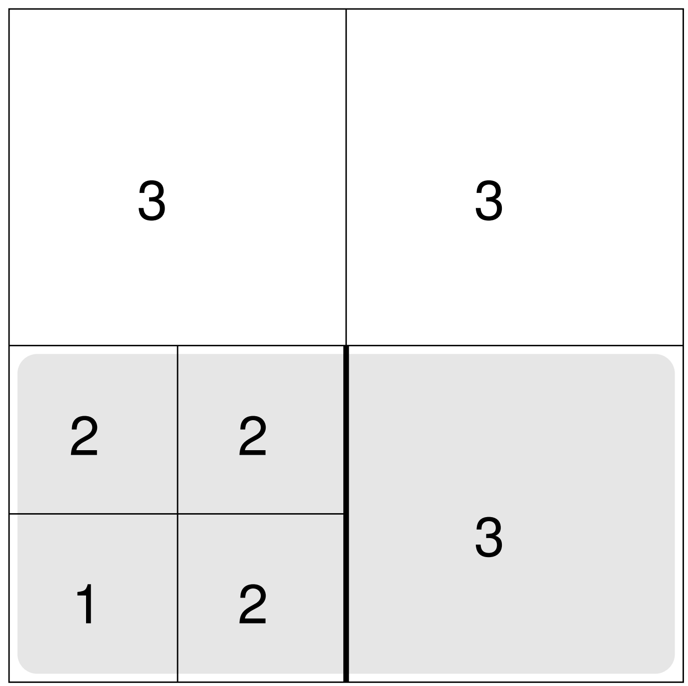


The common edge of K and K' has $p = \min\{p_K, p_{K'}\}$.

- Higher p on an edge than neighbouring elements prescribe is possible!

p Enrichment on Edges

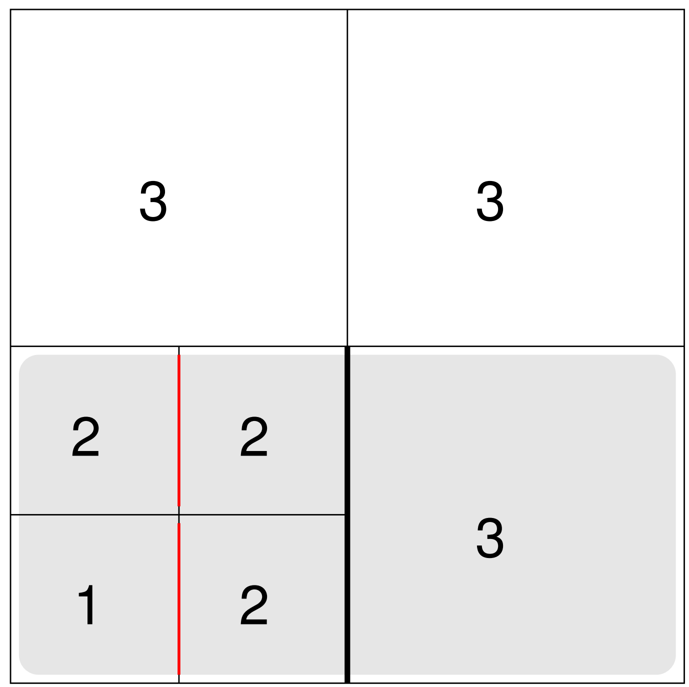
- $p^\star \geq 2$ for the basis functions Φ_i on the marked edge must be possible to achieve exponential convergence.



Analogly for edge and faces in 3D.

p Enrichment on Edges

- $p^\star \geq 2$ for the basis functions Φ_i on the marked edge must be possible to achieve exponential convergence.
- The basis functions on the red edges contribute to Φ_i $\Rightarrow p \geq p^\star$ must be possible and enforced.



Analogly for edge and faces in 3D.

Trunk Spaces

- Tensor Product Space: p^3 shape functions, internal shape functions have indices
  $i = 2, \ldots, p_\xi$,
  $j = 2, \ldots, p_\eta$ and
  $k = 2, \ldots, p_\zeta$
- Trunk Space: $O(p^2)$ shape functions, internal shape functions have indices
  $i = 2, \ldots, p_\xi - 4$,
  $j = 2, \ldots, p_\eta - 4$ and
  $k = 2, \ldots, p_\zeta - 4$ where $i + j + k = 6, \ldots, \max\{p_\xi, p_\eta, p_\zeta\}$, [2].

[2] Szabó and Babuška, “Finite Element Analysis”, John Wiley & Sons, 1991.

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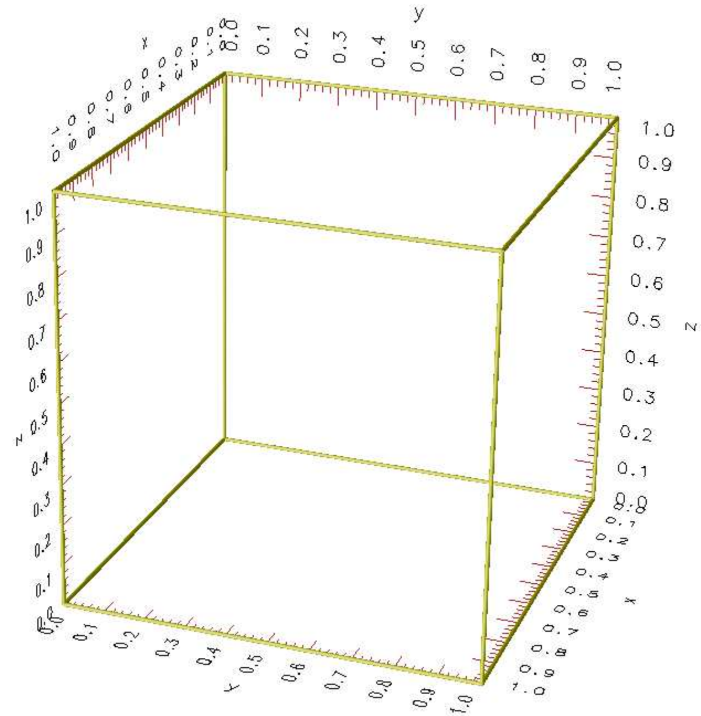
Some Basis Functions



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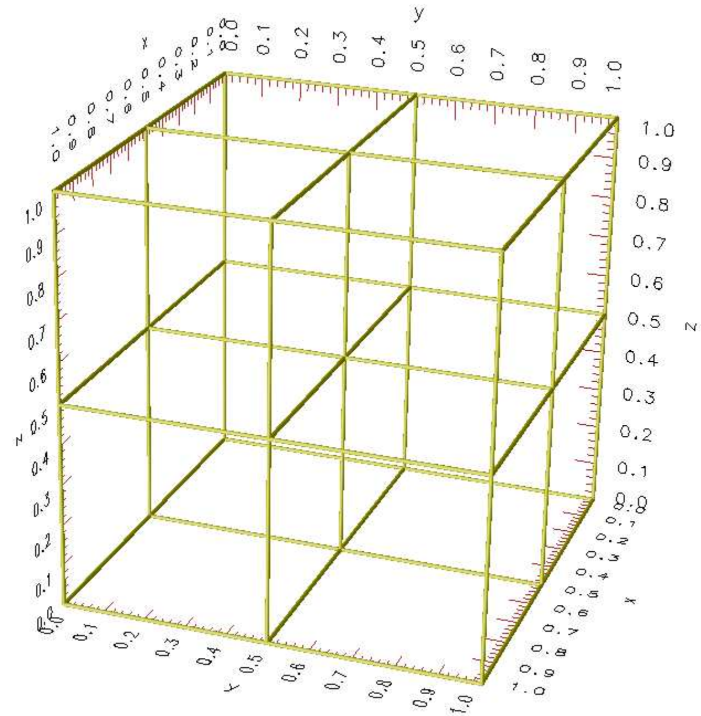

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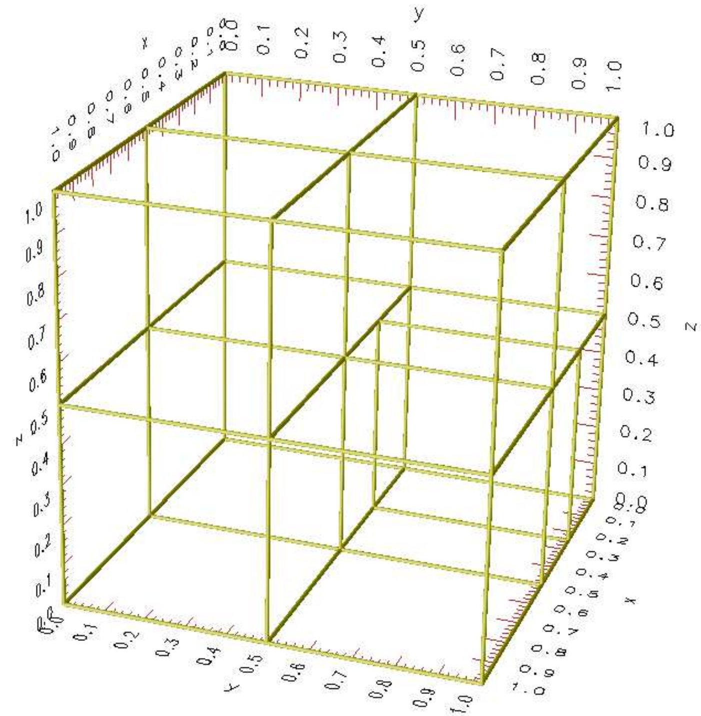

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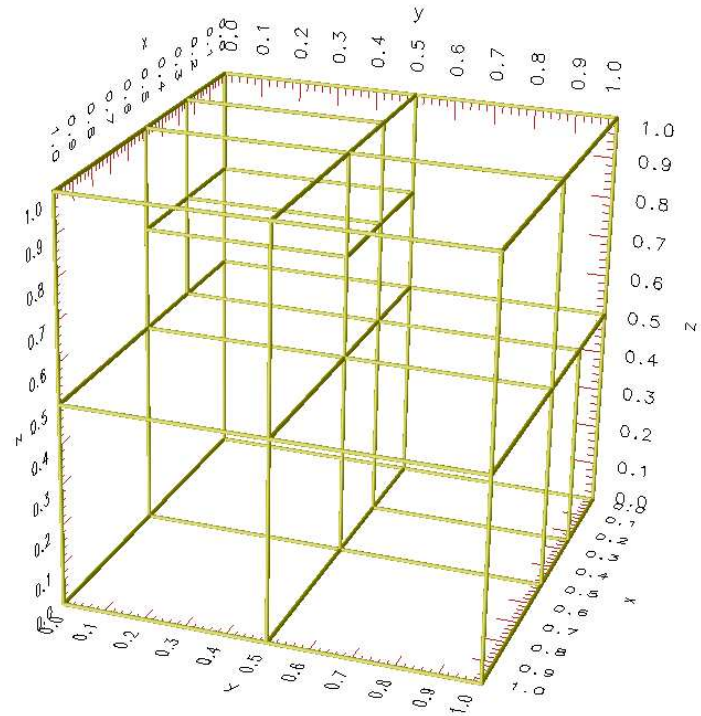

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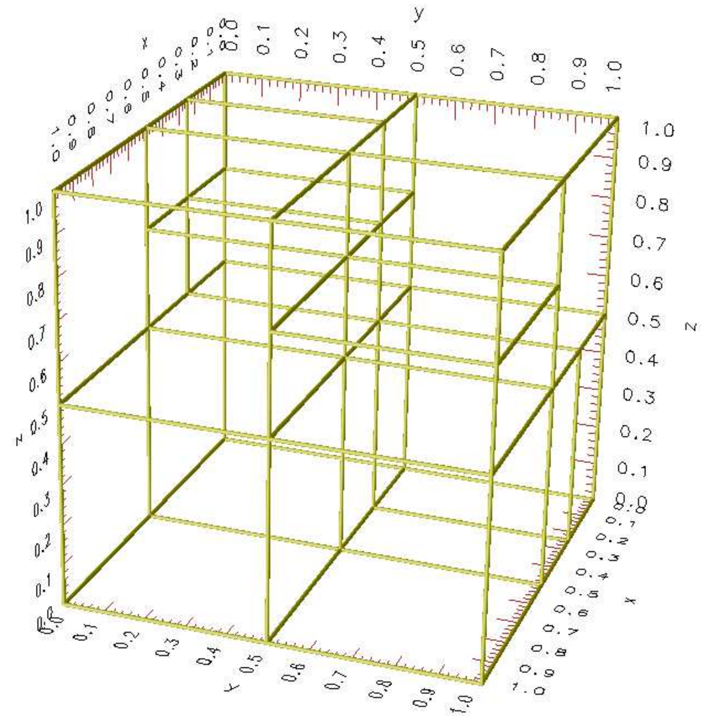
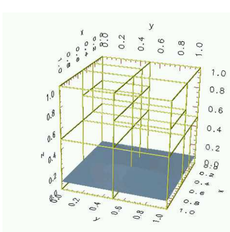
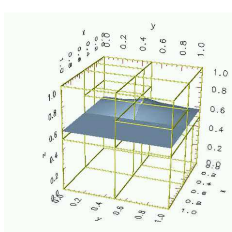
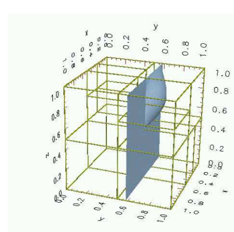


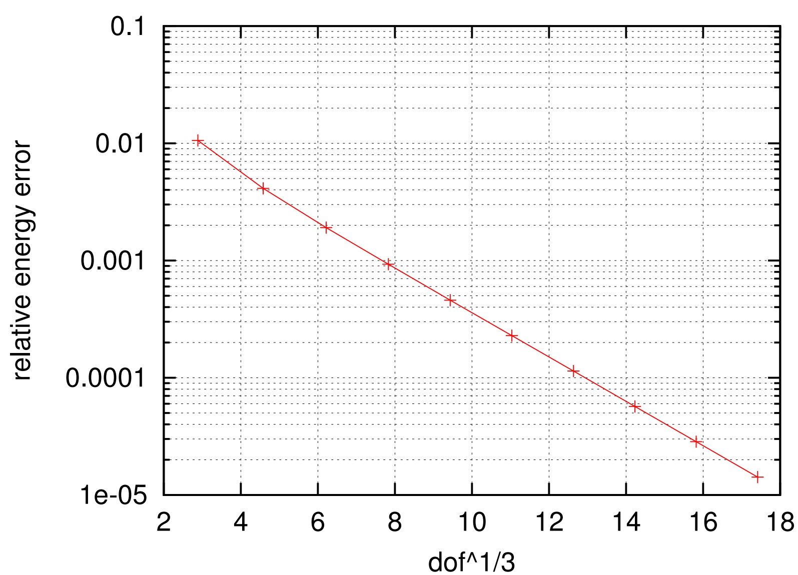


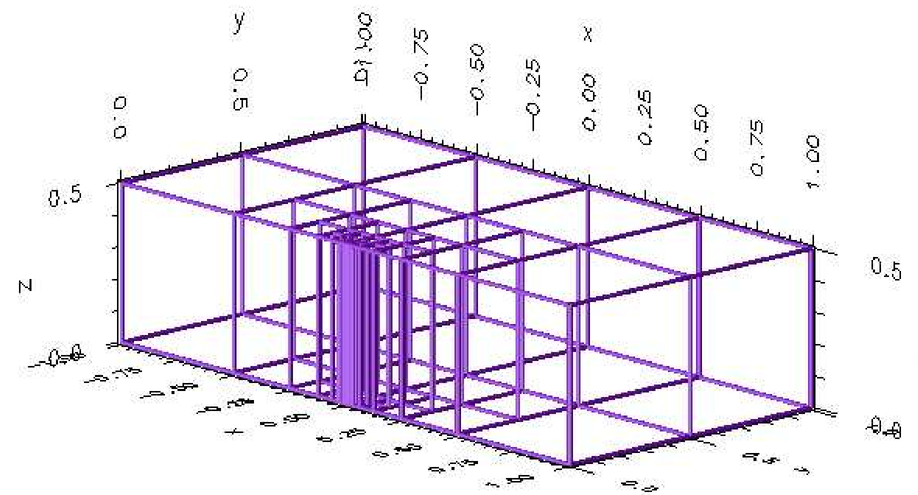




Exponential Convergence in Pseudo-3D

Edge type singularity.





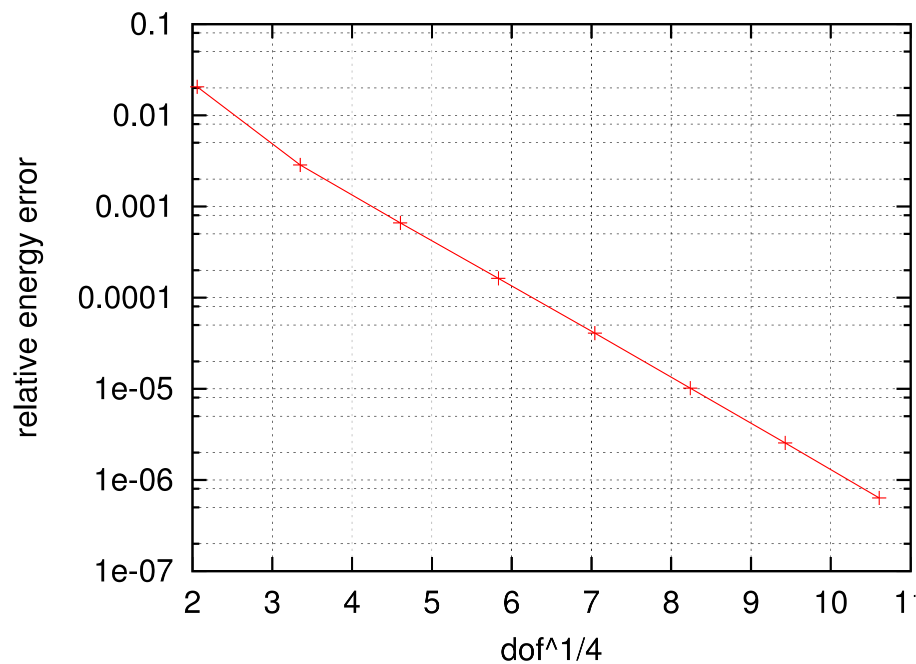
$$-\Delta u + u = f \text{ in } \Omega = (-1,1) \times (0,1) \times (0, 1/2)$$

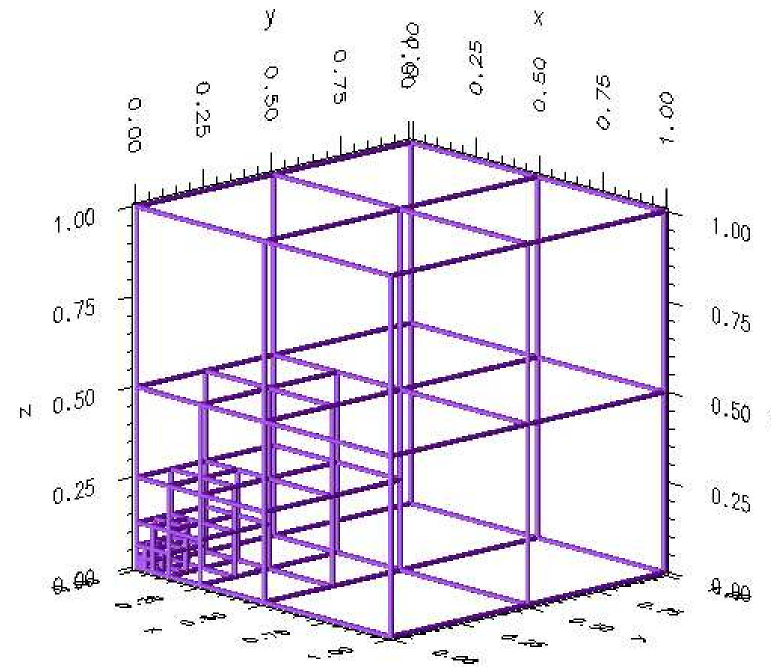
$u(r,\phi,z) = \sqrt{r}\sin(\phi/2)z(1-z)$ in Ω

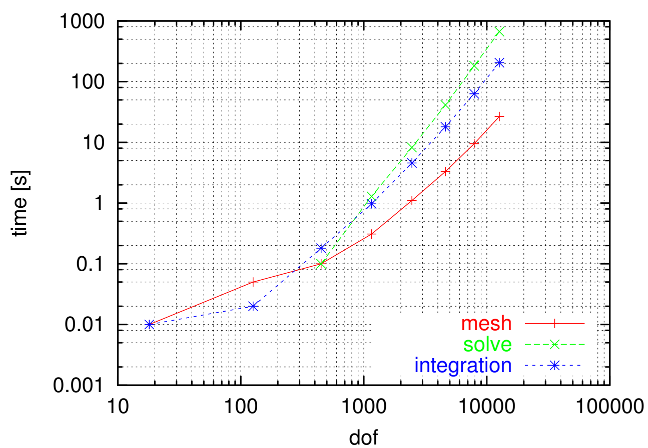
$u = 0$ on $\{z=0\} \subset \partial\Omega$

and on $\{y=0\} \cap \{x \geq 0\} \subset \partial\Omega$

Exponential Convergence in 3D







Vertex type singularity.

$$-\Delta u + u = f \text{ in } \Omega = (0,1)^3$$

$$u(r, \theta, \phi) = \sqrt{r} \sin\theta \sin\phi \qquad \text{in } \Omega$$

$$u = 0 \qquad \text{on } \{y = 0\} \subset \partial\Omega$$

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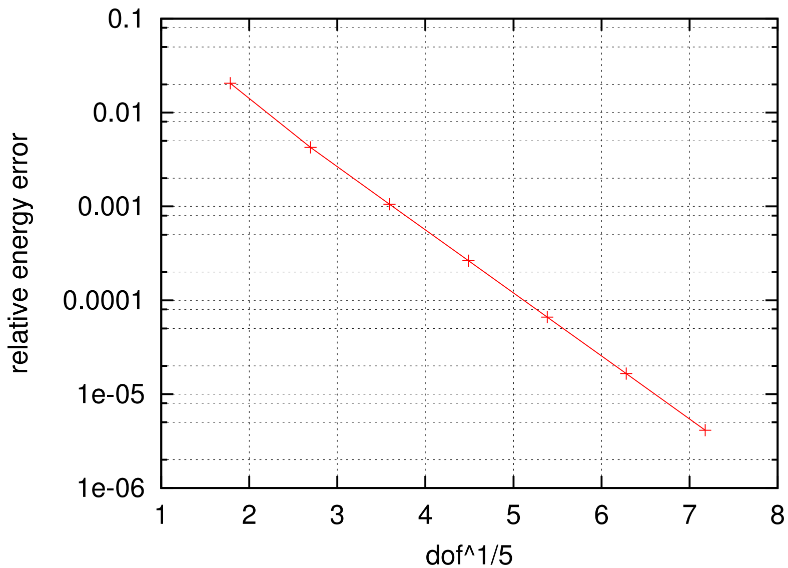
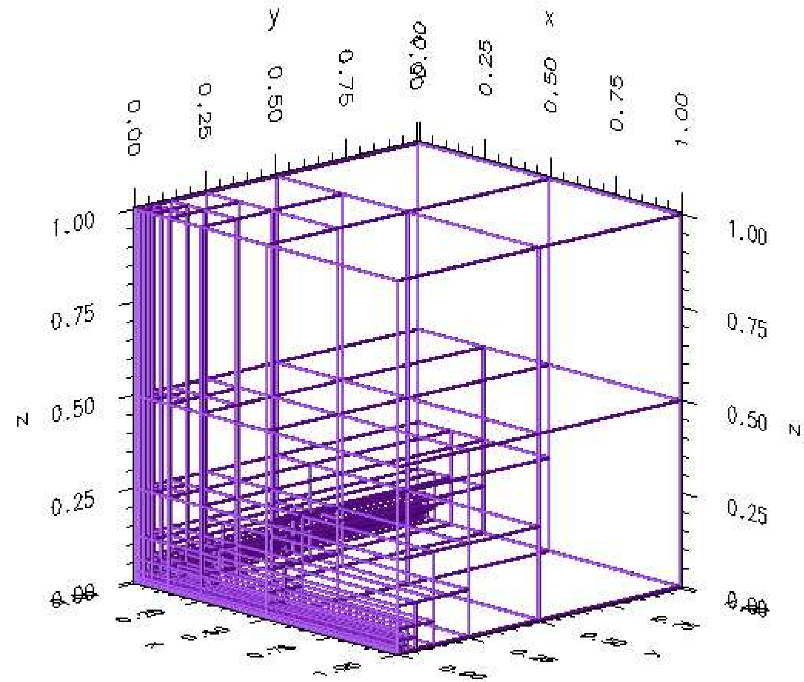
Exp. Conv. in 3D, Edge Mesh





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$$u(r,\theta,\phi) = \sqrt{r}\sin\theta\sin\phi \qquad \text{in } \Omega$$

$$u = 0 \qquad \text{on } \{y=0\} \subset \partial\Omega$$

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Maxwell EVP

Find Eigenvalues $\lambda = \omega^2$ such that $\exists(\underline{E}, \underline{H}) \neq 0$ satisfying

$$\operatorname{curl} \underline{E} - i\omega\mu\underline{H} = 0 \quad \text{and} \quad \operatorname{curl} \underline{H} + i\omega\varepsilon\underline{E} = 0 \quad \text{in } \Omega,$$

with perfect conductor b.c. $\underline{E} \times \underline{n} = 0$, $\underline{H} \cdot \underline{n} = 0$ on $\partial\Omega$. $\underline{E} \in H_0(\mathrm{curl}; \Omega)$.

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"Electric" variational form:
Find the frequencies $\omega > 0$ such that $\exists \underline{E} \in H_0(\mathrm{curl}; \Omega) \setminus \{0\}$ with

$$\int_\Omega {}^1\!/\!_\mu \operatorname{curl} \underline{E} \cdot \operatorname{curl} \underline{F} = \omega^2 \int_\Omega \varepsilon \underline{E} \cdot \underline{F} \text{ and } \operatorname{div} \varepsilon \underline{E} = 0 \quad \forall \underline{F} \in H_0(\mathrm{curl}; \Omega).$$

Weighted Regularization for Maxwell EVP

Find the frequencies $\omega > 0$ such that $\exists \underline{u} \in X_N$ with

$$\int_\Omega \operatorname{curl} \underline{u} \cdot \operatorname{curl} \underline{v} + \langle \underline{u}, \underline{v} \rangle_Y = \omega^2 \int_\Omega \underline{u} \cdot \underline{v}$$

$$\forall \underline{v} \in X_N := \{ \underline{u} \in H_0(\mathrm{curl}; \Omega) : \operatorname{div} \underline{u} \in L^2(\Omega) \}$$

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$$\langle \underline{u}, \underline{v} \rangle_Y = s \int_\Omega \rho(\underline{x}) \operatorname{div} \underline{u} \operatorname{div} \underline{v}$$

Properly chosen weight $\rho(\underline{x})$ and $s \in \mathbb{R}_+$.

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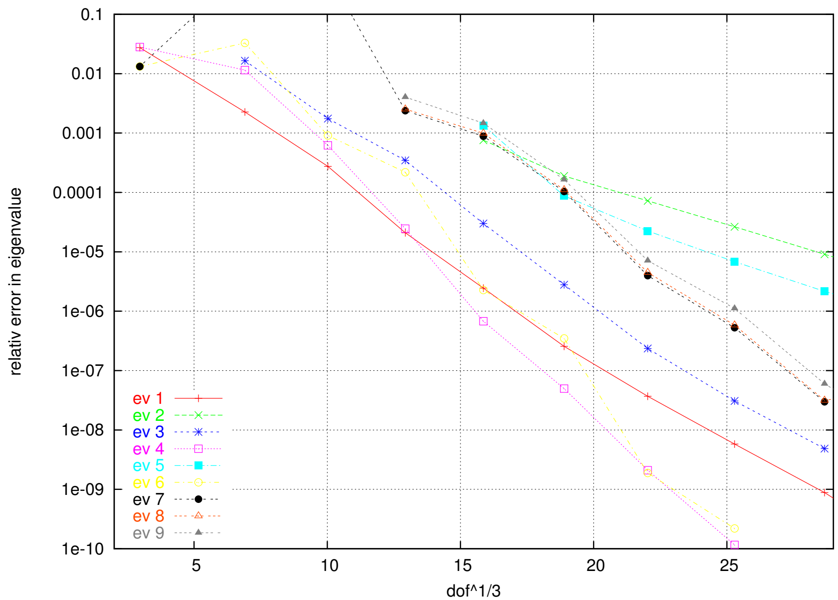
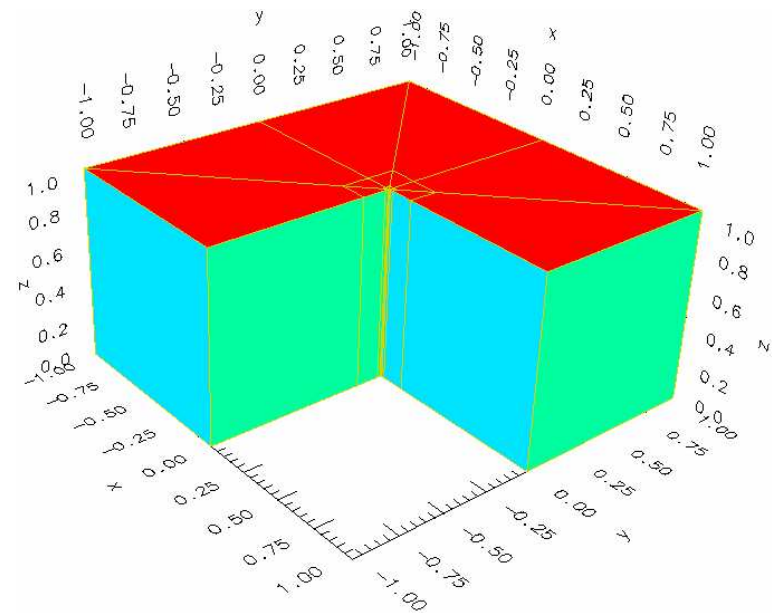
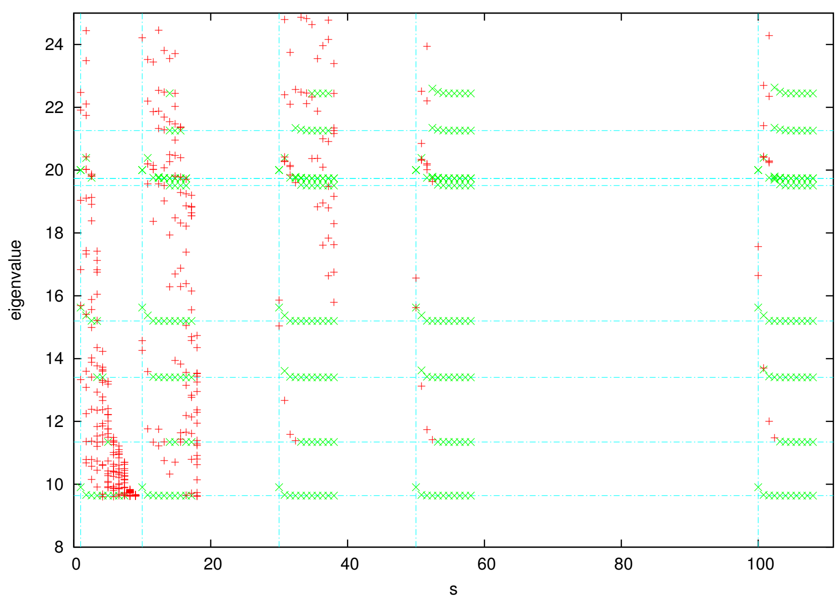
Properly chosen weight $\rho(\underline{x})$ and $s \in \mathbb{R}_+$. Good choice: $\rho(\underline{x}) = r^\alpha$ where r is the distance to a reentrant corner and $\alpha \geq 0$ in a range depending on the angle of the reentrant corner.

[3] Martin Costabel and Monique Dauge, "Weighted regularization of Maxwell equations in polyhedral domains", *Numer. Math.* 93 (2), pp. 239–277 (2002).

EVP in the Thick L Shaped Domain



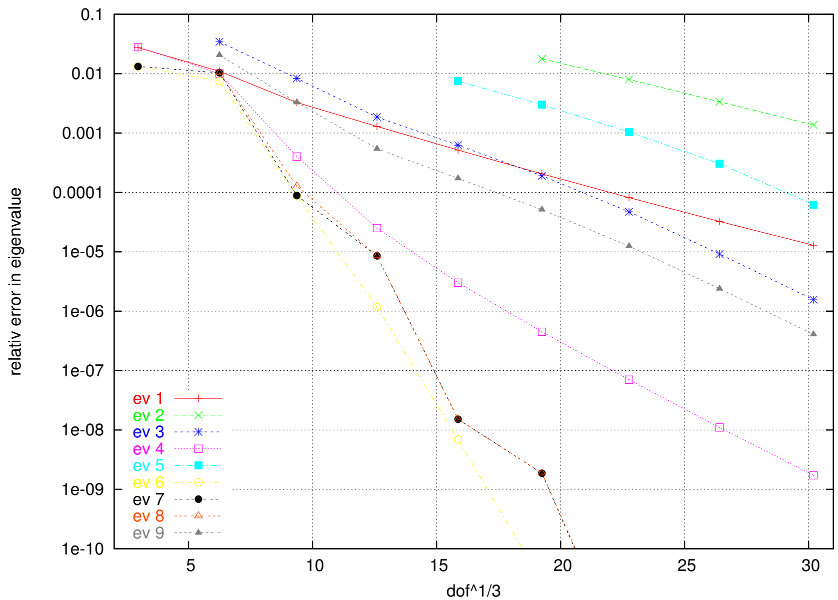


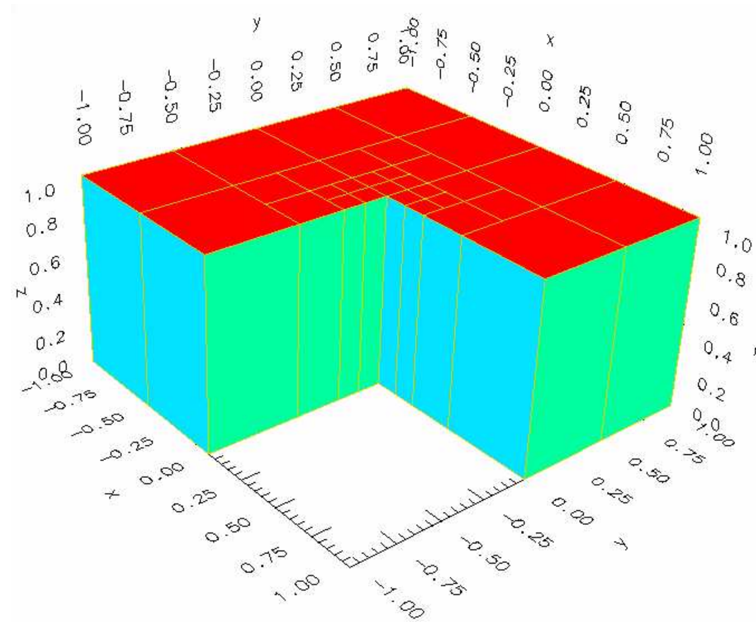


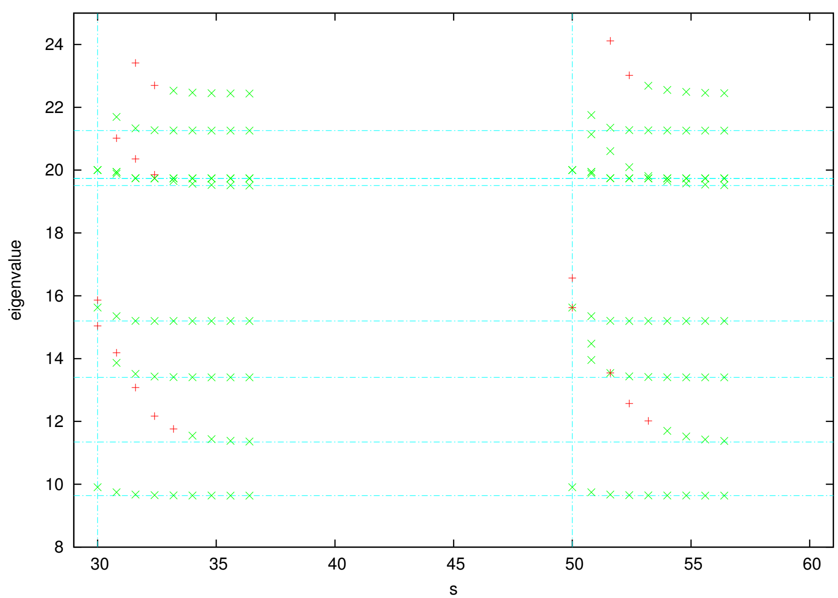
$$\sigma = 0.15$$

$$\alpha = 2$$

EVP in the Thick L Shaped Domain







$$\sigma = 0.5$$

$$\alpha = 2$$

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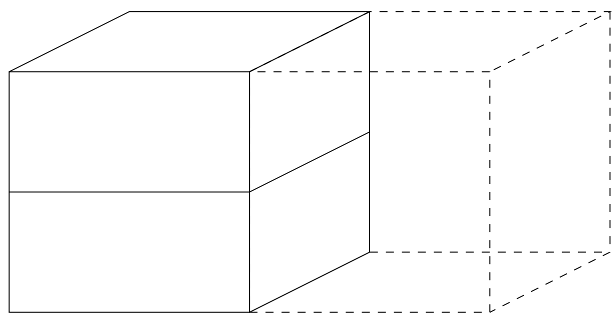
Perspectives

- Maxwell EVP in the Fichera corner
- Anisotropic error estimation,
  anistropic regularity estimation
- Improved mesh handling



- Iterative multilevel domain decompositioning solvers:
  Toselli (Zürich), Schöberl (Linz)
- Stochastic Eigenvalue Problems (e.g. stochastic ε and μ for Maxwell)

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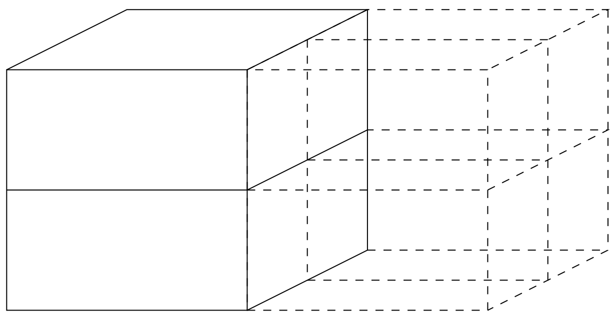

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Hanging Nodes in Isotropic Meshes

- Traverse all cells on locally finest level: mark every vertex / edge / face being used.
- On next (hierarchical) traversal of the mesh:
  - Add dofs which are marked to be on the current level to the list L of local dofs. Mark dof as registered.
  - If cell is on finest level $L \to$ T matrix
  - Otherwise $S \cdot L$ is added to L of child (next deeper level)

Mortar

- Give up $\mathcal{C}^0$, introduce Lagrange multiplier (the mortar)
- $-\Delta u = f$ in Ω with hom. Dirichlet bc. using mortar method leads to
  $\begin{pmatrix} \boldsymbol{A} & \boldsymbol{\Lambda} \\ \boldsymbol{\Lambda}^\top & 0 \end{pmatrix} \cdot \begin{pmatrix} \underline{u} \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}$, ie.

  SPD PDE $\not\Rightarrow$ SPD matrix

  $\Rightarrow$ conjugate gradients not applicable

  $\Rightarrow$ no standard domain decompositioning solvers

  $\Rightarrow$ inf-sup condition needed

- The inf-sup cond. is OK in 2D, 3D for shape regular meshes.
  Not OK for hp FEM, existing proofs only for uniform meshes.
- Analogly for Discontinous Galerkin in 3D: Stability of hp DG on geometric meshes is not clear. First results by Schwab, Toselli, Schötzau for Stokes (not Mortar).

Shape Functions

The reference element shape functions on $(-1,1)$ of order p [4]:

$$N_i(\xi) = \begin{cases} \frac{1-\xi}{2} & i = 0 \\ \frac{1-\xi}{2}\frac{1+\xi}{2}P_{i-1}^{1,1}(\xi) & 1 \leq i \leq p-1 \\ \frac{1+\xi}{2} & i = p \end{cases}$$

$P_{i-1}^{1,1}(\xi)$ are integrated Legendre Polynomials: $L_i(\xi) = P_i^{0,0}(\xi)$ and

$$\int_{-1}^{\xi}(1-x)^{\alpha}(1+x)^{\beta}P_i^{\alpha,\beta}(x)\,dx = \frac{-1}{2i}(1-\xi)^{\alpha+1}(1+\xi)^{\beta+1}P_{i-1}^{\alpha+1,\beta+1}(\xi)$$

$$\Rightarrow \int_{-1}^{\xi}P_i^{0,0}(x)\,dx = \frac{-1}{2i}(1-\xi)(1+\xi)P_{i-1}^{1,1}(\xi)$$

[4] Karniadakis and Sherwin, “Spectral/hp Element Methods for CFD”, Oxford University Press, 1999.

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