Anisotropic h and p refinement for conforming FEM in 3D

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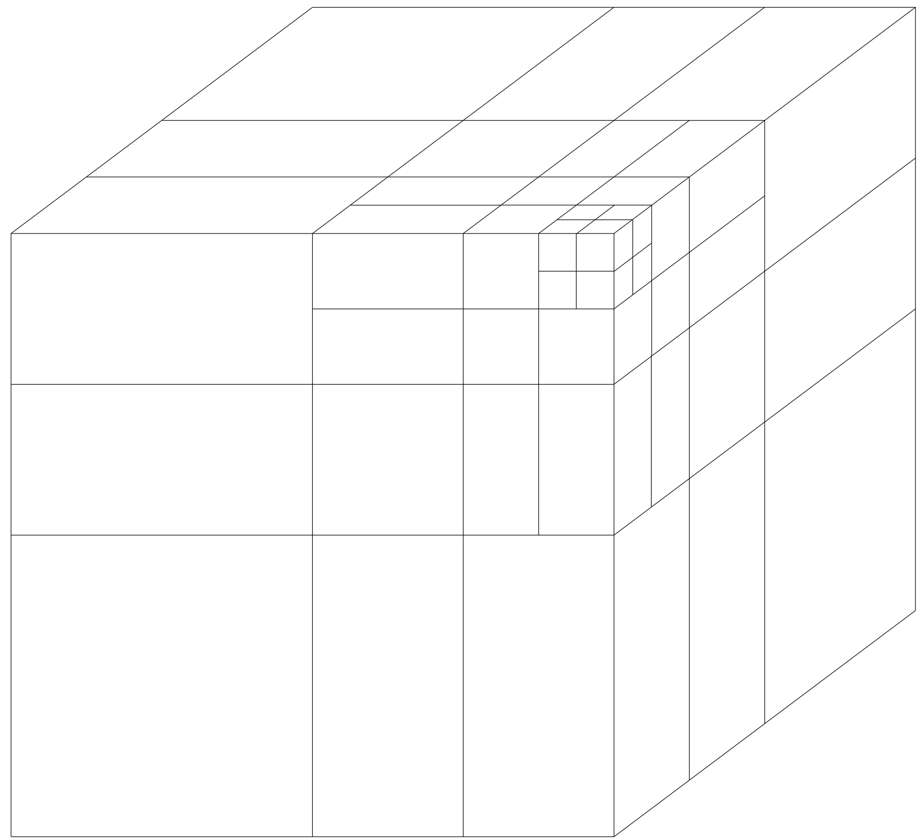
Goal



- Hierarchy of hanging nodes
- Anisotropic refinements

Overview

- Introduction
- Anisotropic h refinements
  - S and T matrices
  - Assembly of Supports

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- hp Meshes
- Perspectives

Previous hp Software

- Szabó 1985: PROBE (p only)
- Demkowicz, Oden, Rachowicz et al. 1989: PHLEX, hp90
- Anderson: STRIPE (p only on a-priori generated meshes)
- Flaherty, Shephard: Tetrahedra only (3D anisotropy?)
- Karniadakis, Sherwin: NEKTAR (regular meshes only, tetrahedra, hexahedra, prisms)
- Devloo
- Szabó since 1995: STRESSCHECK
- Heuveline et al.: HiFlow

FE Method

- Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ (dimension independent design)
- Find $u \in V$ such that

$$a(u, v) = l(v) \quad \forall v \in V,$$

  V a FE space, $a(.\,,.)$ a bilinear form and $l(.)$ a linear form.
- Standard FE: $V \subset H^1(\Omega)$

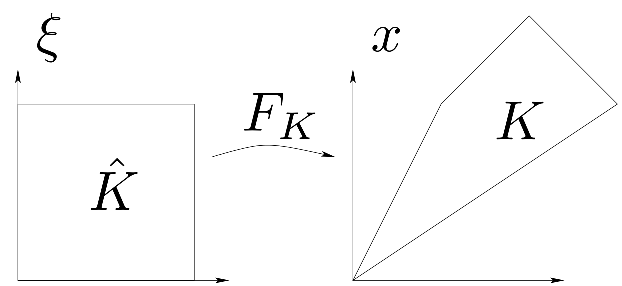
$$\begin{aligned} V &= S^{1,\underline{p}}(\Omega, \mathcal{T}) \\ &= \left\{ u \in H^1(\Omega) : u|_K \circ F_K \in \mathcal{Q}_p \; \forall K \in \mathcal{T} \right\} \end{aligned}$$

  $\Rightarrow u \in V$ is continuous, ie. $\mathcal{C}^0(\bar{\Omega})$.
- Vector valued problems are possible

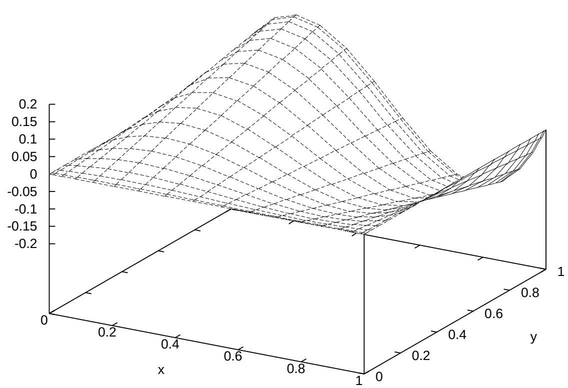
FE Space: Generalities

- Basis $\{\Phi_i\}_{i=1}^{N}$ constructed from element shape functions ϕ_j^K on elements $K \in \mathcal{T}$.
- Reference element shape functions: N_j, element map: $F_K : \hat{K} \to K$

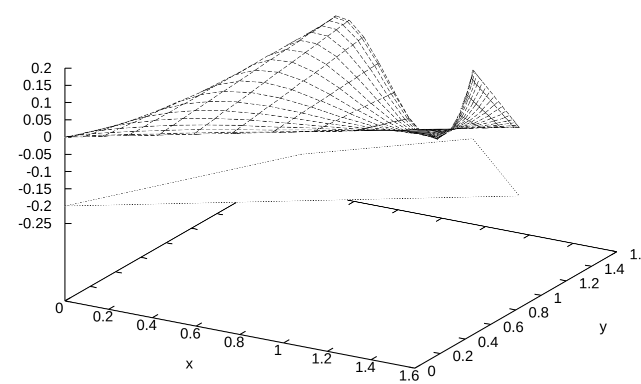
$$\Rightarrow \phi_j^K \circ F_K = N_j .$$



N_j

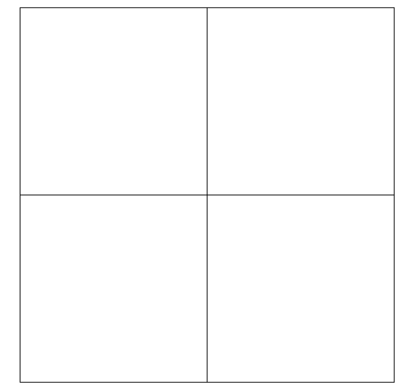


Φ_j



FE Meshes

Local refinements as mean to improve approximation of exact solution by FE solution



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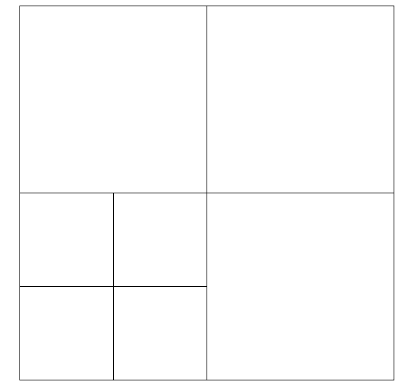

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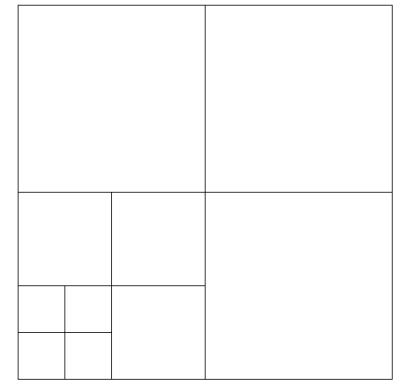

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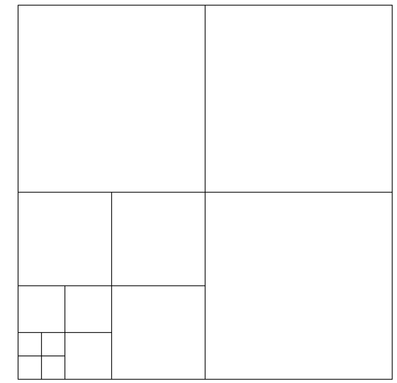

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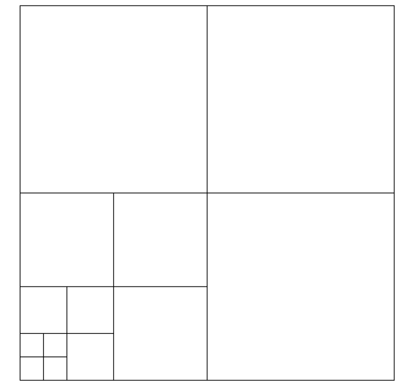
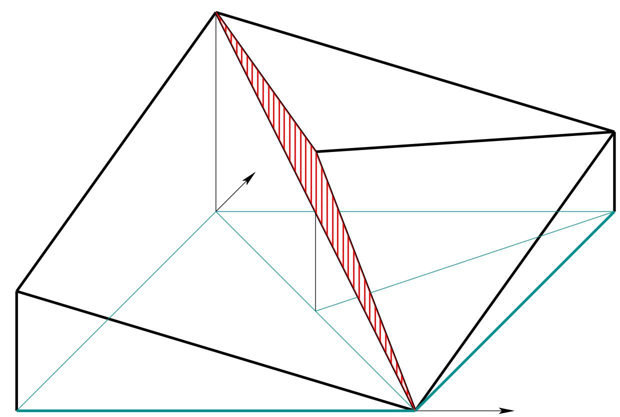
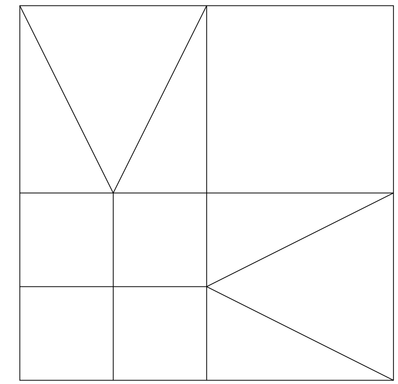


Local refinements as mean to improve approximation of exact solution by FE solution



But standard FE forbids locally refined grids: discontinuities are possible.

Mortar vs. Enforcing Continuity



- Topolocigal closure

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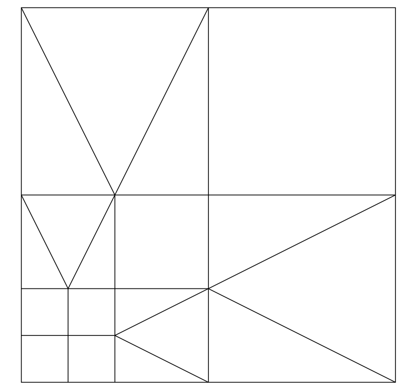

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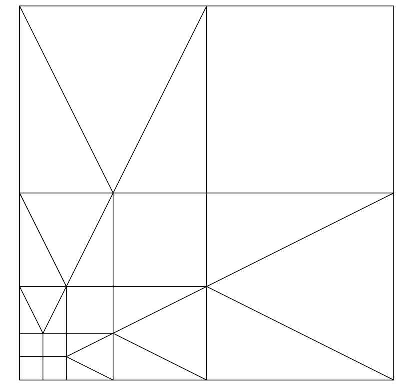

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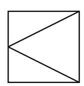
  Drawbacks: more elements, more element types, what about refining a  ?

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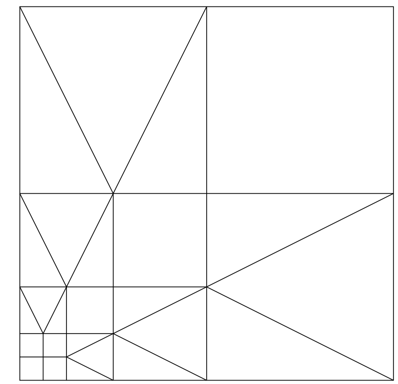

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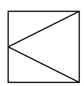
  Drawbacks: more elements, more element types, what about refining a  ?

- Our philosophy: hexahedral meshes only (tensorized interpolants, spectral quadrature techniques)

- Our solution: Treating the constraints induced by the hanging nodes
  Why conforming? $a(u, v) = a(v, u)$ and $a(u, u) \geq \alpha \|u\|_V^2 \Rightarrow \boldsymbol{A}$ SPD, pccg . . .

Our Software: Concepts

- Started by Christian Lage during his Ph.D. studies (1995).
- Used and improved by Frauenfelder, Matache, Schmidlin, Schmidt and several students.
- Concept Oriented Design using mathematical principles.
- Currently two parts: hp-FEM, BEM (wavelet and multipole methods).
- C++

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- Anisotropic h refinements
  - **S and T matrices**
  - Assembly of Supports
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T Matrix

Definition 1 (T Matrix). *Element shape functions* $\{\phi_j^K\}_{j=1}^{m_K}$ *on element* K, *global basis functions* $\{\Phi_i\}_{i=1}^{N}$.
The T matrix $\boldsymbol{T}_K \in \mathbb{R}^{m_K \times N}$ *of element* K *is implicitly defined by*

$$\Phi_i|_K = \sum_{j=1}^{m_K} [\boldsymbol{T}_K]_{ji}\, \phi_j^K$$

as vectors:

$$\underline{\Phi}|_K = \boldsymbol{T}_K^\top \underline{\phi}^K.$$

Assembly using T Matrices

Assembling:

$$\underline{l} = l(\underline{\Phi}) = l\Big(\sum_{\tilde{K}} \boldsymbol{T}_{\tilde{K}}^{\top} \underline{\phi}^{\tilde{K}}\Big) = \sum_{\tilde{K}} \boldsymbol{T}_{\tilde{K}}^{\top} l(\underline{\phi}^{\tilde{K}}) = \sum_{\tilde{K}} \boldsymbol{T}_{\tilde{K}}^{\top} \underline{l}_{\tilde{K}}$$

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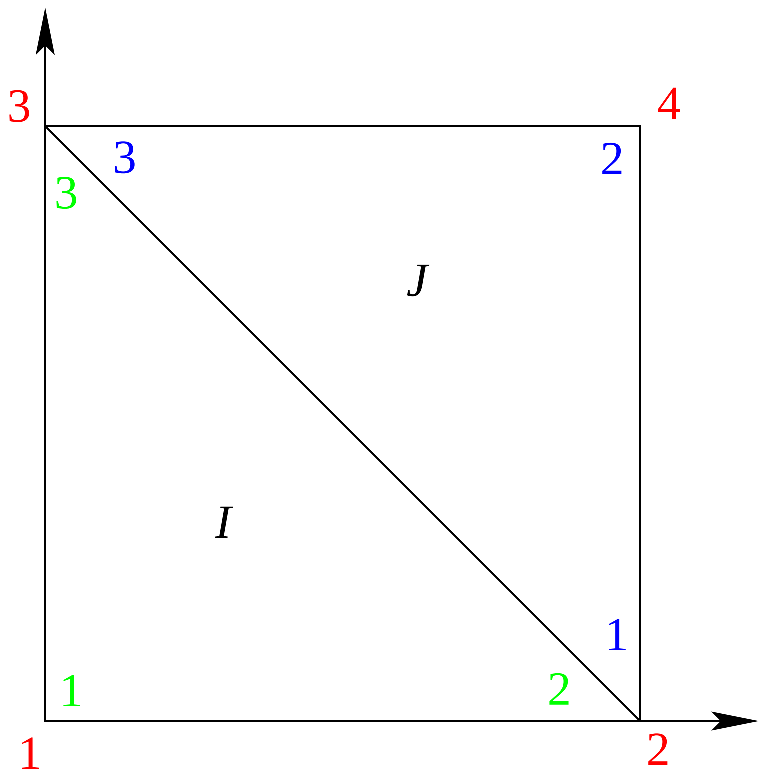
$$\boldsymbol{A} = a(\underline{\Phi}, \underline{\Phi}) = \sum_{K,\tilde{K}} \boldsymbol{T}_{\tilde{K}}^{\top} a(\underline{\phi}^{K}, \underline{\phi}^{\tilde{K}}) \boldsymbol{T}_K = \sum_{K,\tilde{K}} \boldsymbol{T}_{\tilde{K}}^{\top} \boldsymbol{A}_{\tilde{K}K} \boldsymbol{T}_K$$

Note: $\boldsymbol{A}_{\tilde{K}K} = 0$ in standard FEM for $\tilde{K} \neq K$.

Example: Regular Mesh

Two elements with three local shape functions each and four global basis functions.



$$T_I = \begin{pmatrix} & 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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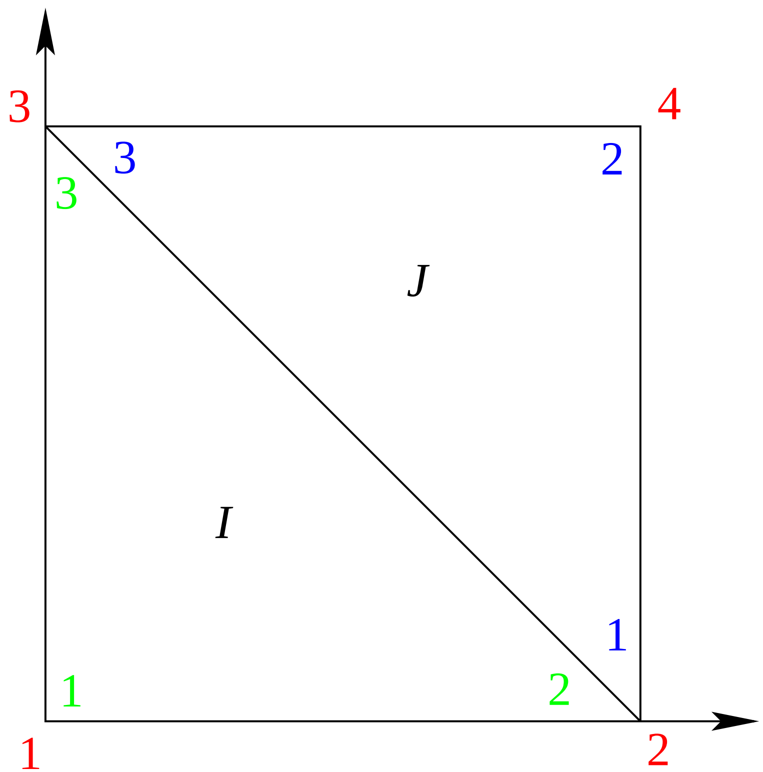
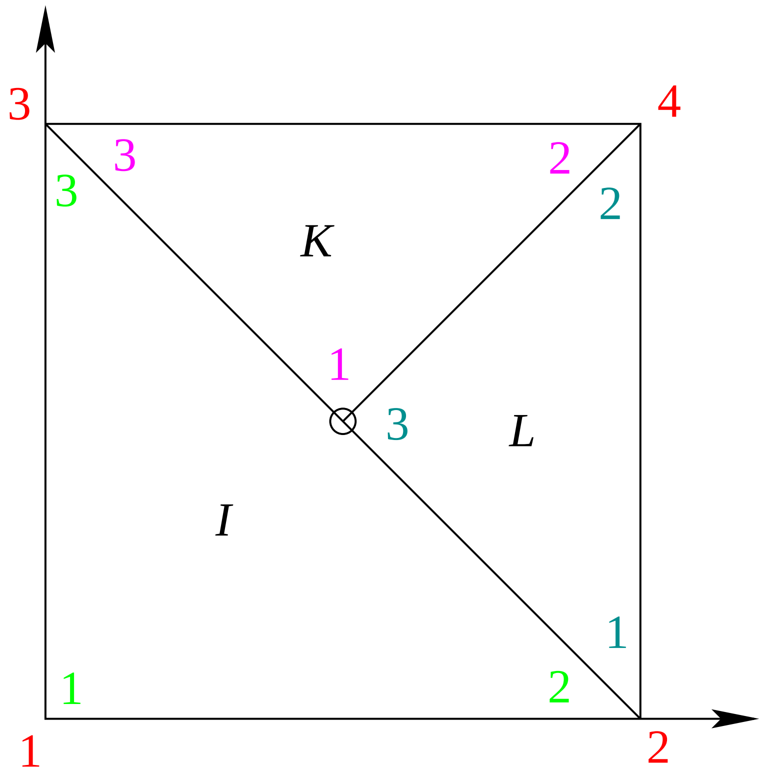


$$\boldsymbol{T}_I = \begin{pmatrix} & 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\boldsymbol{T}_J = \begin{pmatrix} & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Example: Irregular Mesh

Three elements with three local shape functions each and four global basis functions. The hanging node is marked with $\circ$.



$$\boldsymbol{T}_L = \begin{pmatrix} & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

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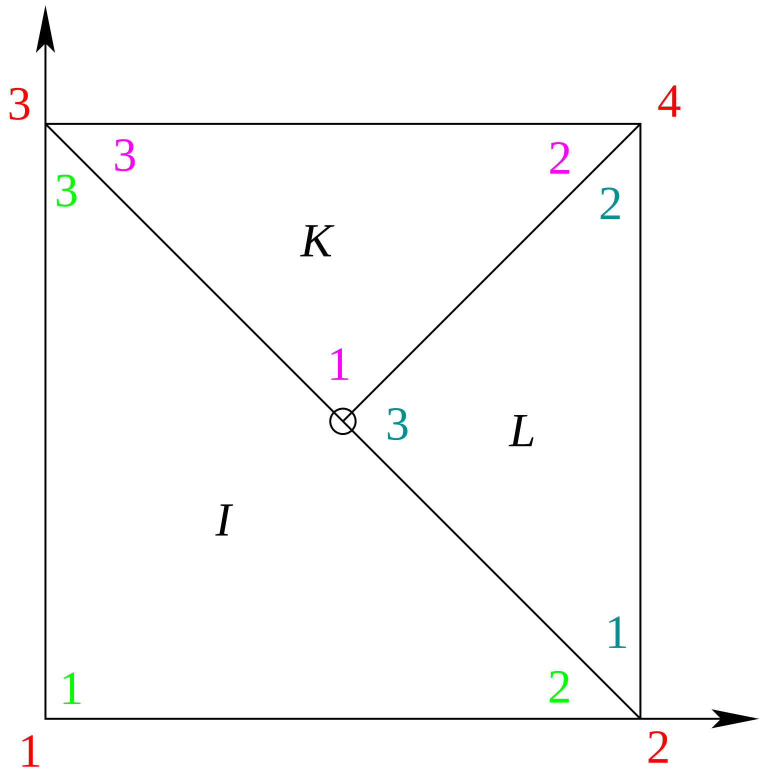

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$$

$$
\boldsymbol{T}_K = \begin{pmatrix} & 1 & 2 & 3 & 4 \\ 1 & 0 & 1/2 & 1/2 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 \end{pmatrix}
$$

$\Rightarrow$ continuous basis functions.

Generation of T Matrices

- **Regular Mesh:** Counting and assigning indices with respect to topological entities such as vertices, edges and faces. Explained in detail later.

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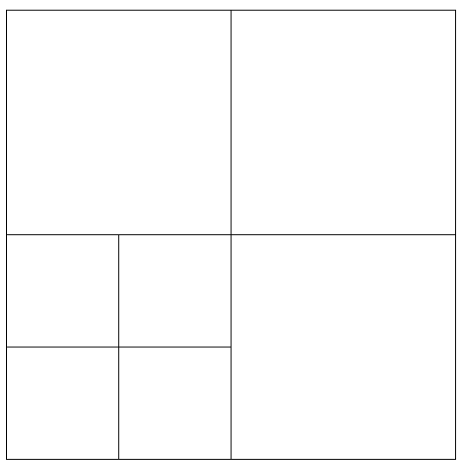
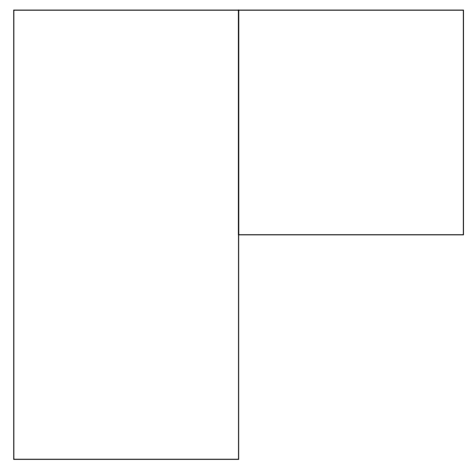
- **Regular Mesh:** Counting and assigning indices with respect to topological entities such as vertices, edges and faces. Explained in detail later.
- **Irregular Mesh:** Irregularity due to a refinement of an initially regular mesh.

  OK: , not OK: 

  Explanation follows.

T Matrices for Irregular Meshes

Irregularity due to a refinement of an initially regular mesh.

Mesh	$\mathcal{M}$	refine	$\mathcal{M}'$
Basis fcts.	$B = B_{\mathrm{repl}} \cup B_{\mathrm{keep}}$	$\longrightarrow$	$B' = B_{\mathrm{ins}} \cup B_{\mathrm{keep}}$

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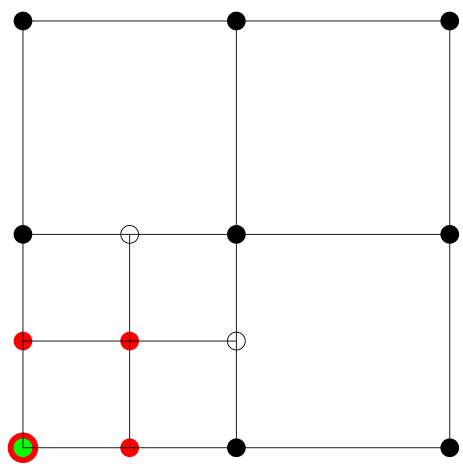

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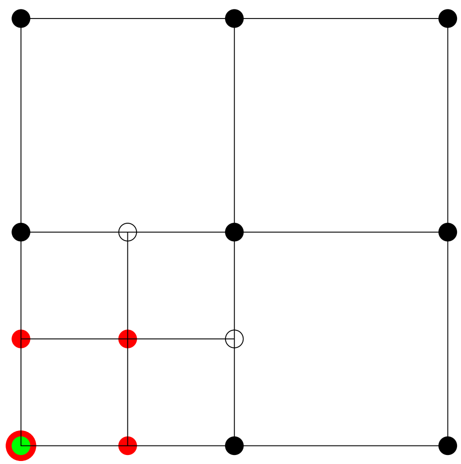

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Every element of B has a column in the T matrix. Generation is

- easy for B_{ins} (like regular mesh),
- simple for B_{keep}: modify column from $\mathcal{M}$ by S matrix.

S Matrix

Definition 2 (S Matrix). *Let $K' \subset K$ be the result of a refinement of element K. The S matrix $\boldsymbol{S}_{K'K} \in \mathbb{R}^{m_{K'} \times m_K}$ is defined by*

$$\phi_j^K\Big|_{K'} = \sum_{l=1}^{m_{K'}} [\boldsymbol{S}_{K'K}]_{lj}\, \phi_l^{K'}$$

as vectors:

$$\underline{\phi}^K\Big|_{K'} = \boldsymbol{S}_{K'K}^\top \underline{\phi}^{K'}$$

$\phi_j^K\big|_{K'}$ is represented as a linear combination of the shape functions $\left\{\phi_l^{K'}\right\}_{l=1}^{m_{K'}}$ of K'.

Application of S Matrix

Proposition 1. *Let $K' \subset K$ be the result of a refinement of an element K. Then, the T matrix of K' can be computed as*

$$\boldsymbol{T}_{K'} = \boldsymbol{S}_{K'K} \boldsymbol{T}_K^{\mathrm{keep}} + \boldsymbol{T}_{K'}^{\mathrm{ins}}$$

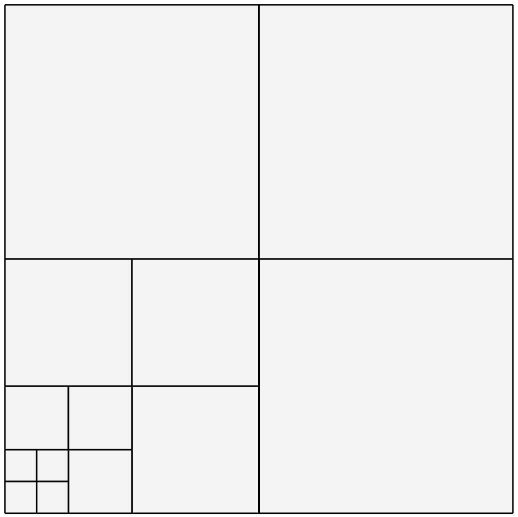
where $\boldsymbol{T}_K^{\mathrm{keep}}$ denotes the T matrix of element K (with columns not related to functions in B_{keep} set to zero) and $\boldsymbol{T}_{K'}^{\mathrm{ins}}$ the T matrix for functions in B_{ins} with respect to K'.

Proposition 2. *Let $\hat{K}' \subset \hat{K}$ be the result of a refinement of the reference element $\hat{K}$ with $H : \hat{K} \to \hat{K}'$ the subdivision map. The element maps are*

$$F_K : \hat{K} \to K \text{ and } F_{K'} : \hat{K} \to K'$$

and $F_{K'} \circ H^{-1} = F_K$ holds. Then, $\boldsymbol{S}_{\hat{K}'\hat{K}} = \boldsymbol{S}_{K'K}$.

Meshes



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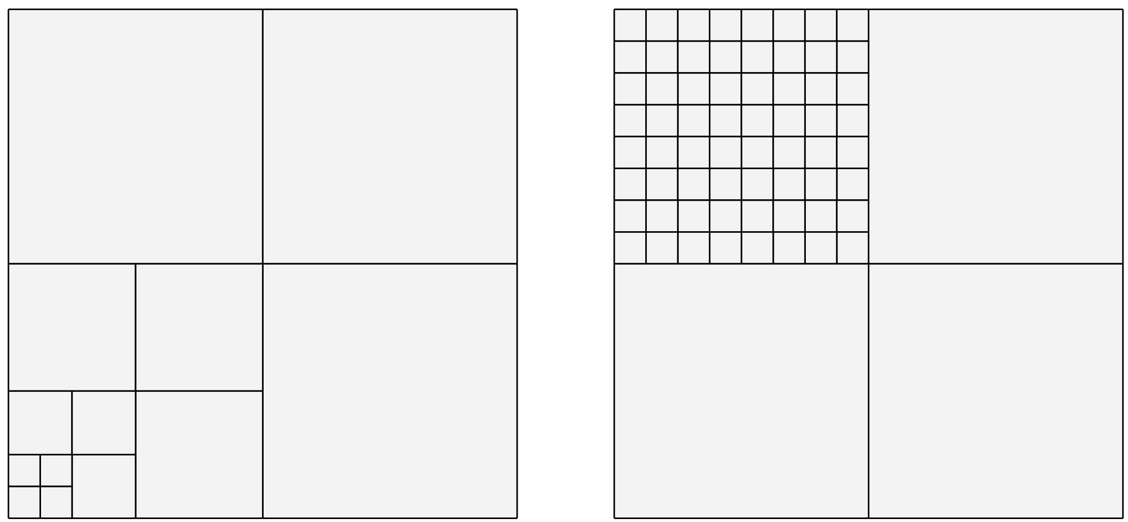

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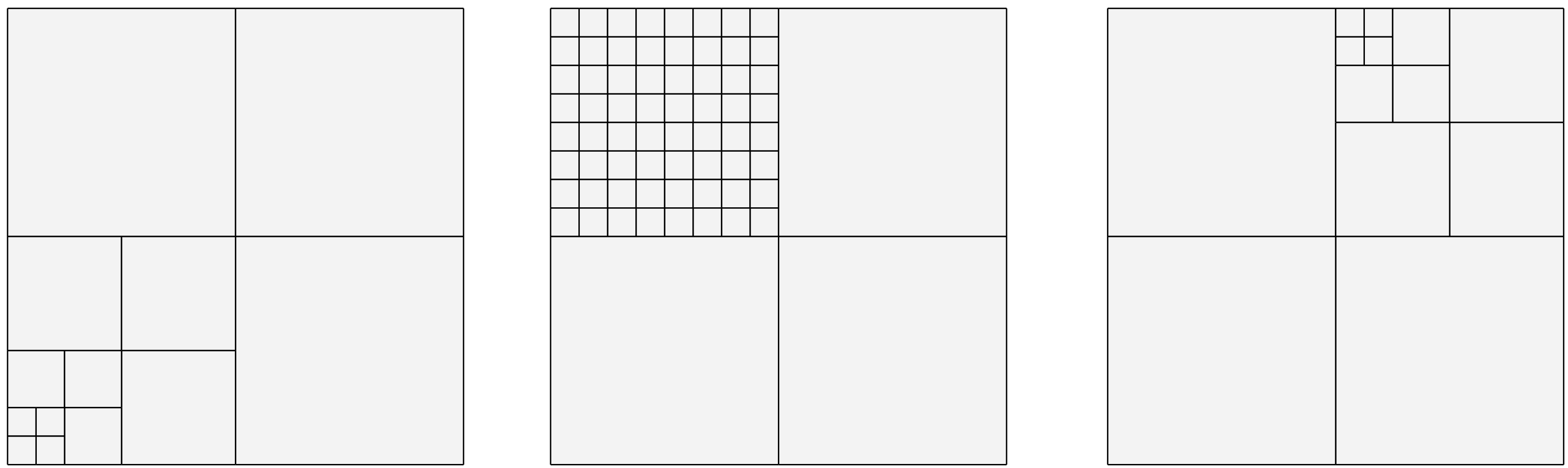

S Matrix in Dimension $d = 1$

Subdividing $\hat{J} = (0,1)$ in $\hat{J}' = (0, 1/2)$ and $\hat{J}^\star = (1/2, 1)$ with the reference element shape functions

$$N_j(\xi) = \begin{cases} 1-\xi & j=1 \\ \xi & j=2 \\ \xi(1-\xi)P^{1,1}_{j-3}(2\xi-1) & j=3,\ldots,J \end{cases}$$

yields (solving a linear system) for $J = 4$:

$$\boldsymbol{S}_{\hat{J}',\hat{J}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 1/4 & 0 \\ 0 & 0 & 1/4 & -3/4 \\ 0 & 0 & 0 & 1/8 \end{pmatrix} \text{ and } \boldsymbol{S}_{\hat{J}^\star,\hat{J}} = \begin{pmatrix} 1/2 & 1/2 & 1/4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 1/8 \end{pmatrix}.$$

Hierarchic shape functions $\Rightarrow$ hierarchic S matrices.

S Matrices: Tensor Product in 2D

- $d > 1$ with hexahedral meshes $\Rightarrow$ S matrices are built from tensor products of 1D S matrices.

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- In 2D: $N_{i,j} = N_i \otimes N_j$, the four bilinear shape functions are:

$$N_{1,2}(\underline{\xi}) = N_1(\xi_1) \cdot N_2(\xi_2) \qquad N_{2,2}(\underline{\xi}) = N_2(\xi_1) \cdot N_2(\xi_2)$$

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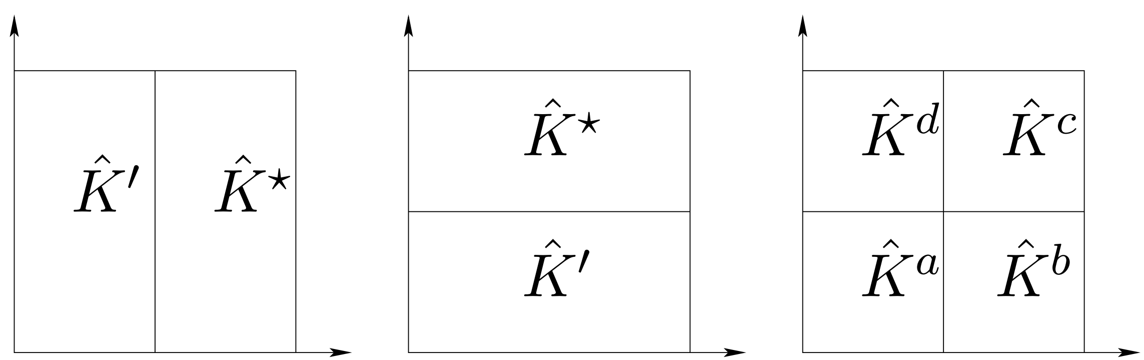
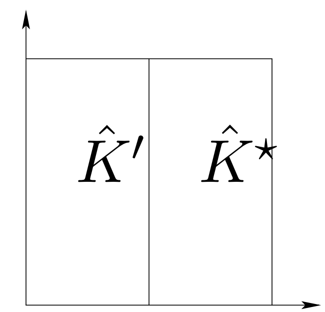
- Consider the subdivisions:



S Matrices: Tensor Product in 2D II

Subdivision map of left variant: $H : \hat{K} \to \hat{K}', \underline{\xi} \mapsto \begin{pmatrix} \xi_1/2 \\ \xi_2 \end{pmatrix}$. S matrix $\boldsymbol{S}_{\hat{K}'\hat{K}}$ is defined by:

$$N_{i,j}|_{\hat{K}'} = \sum_{k,l} \left[\boldsymbol{S}_{\hat{K}'\hat{K}}\right]_{(k,l),(i,j)} N_{k,l} \circ H^{-1}.$$



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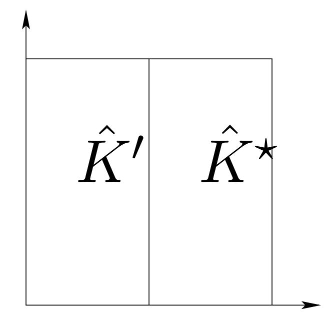

Tensor product shape functions:

$$(N_i \otimes N_j)|_{\hat{K}'} = \sum_{k,l} \left[\boldsymbol{S}_{\hat{K}'\hat{K}}\right]_{(k,l),(i,j)} (N_k \otimes N_l) \circ H^{-1}. \quad (1)$$

S Matrices: Tensor Product in 2D III

S matrices for 1D reference element shape fcts. used in (1):

$$N_i|_{\hat{J}'} = \sum_m [\boldsymbol{S}_{\hat{J}',\hat{J}}]_{mi} N_m \circ G^{-1}$$ for the ξ_1 part and

$$N_j = \sum_n [\boldsymbol{E}]_{nj} N_n$$ for the ξ_2 part,

where $G : \xi \mapsto \xi/2$.

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$$N_j = \sum_n [\boldsymbol{E}]_{nj} N_n$$ for the ξ_2 part,

where $G : \xi \mapsto \xi/2$. Plugging into the left hand side of (1) yields:

$$\begin{aligned}(N_i \otimes N_j)|_{\hat{K}'} = N_i|_{\hat{J}'} \otimes N_j &= \sum_{m,n} \left(\left[\boldsymbol{S}_{\hat{J}',\hat{J}}\right]_{mi} N_m \circ G^{-1}\right) \otimes \Big([\boldsymbol{E}]_{nj} N_n\Big) \\ &= \sum_{m,n} \left[\boldsymbol{S}_{\hat{J}',\hat{J}}\right]_{mi} \cdot [\boldsymbol{E}]_{nj} N_m \circ G^{-1} \otimes N_n.\end{aligned}$$

S Matrices: Tensor Product in 2D IV

Comparing with the right hand side of (1):

$$\sum_{m,n} \left[\boldsymbol{S}_{\hat{J}',\hat{J}}\right]_{mi} \cdot [\boldsymbol{E}]_{nj}\, N_m \circ G^{-1} \otimes N_n$$

$$= \sum_{k,l} \left[\boldsymbol{S}_{\hat{K}'\hat{K}}\right]_{(k,l),(i,j)} N_k \circ G^{-1} \otimes N_l .$$

S Matrices: Tensor Product in 2D IV

Comparing with the right hand side of (1):

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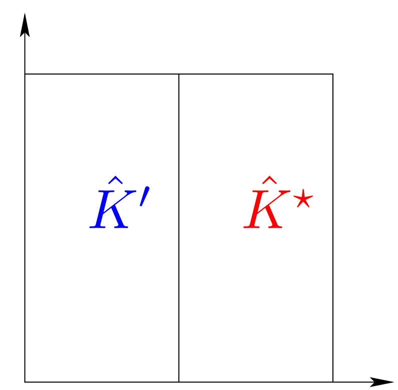
$$= \sum_{k,l} \left[\boldsymbol{S}_{\hat{K}'\hat{K}}\right]_{(k,l),(i,j)} N_k \circ G^{-1} \otimes N_l.$$

Therefore for the vertical subdivision:

$\boldsymbol{S}_{\hat{K}'\hat{K}} = \boldsymbol{S}_{\hat{J}'\hat{J}} \otimes \boldsymbol{E}$ for the left quad $\hat{K}'$,

$\boldsymbol{S}_{\hat{K}^\star\hat{K}} = \boldsymbol{S}_{\hat{J}^\star\hat{J}} \otimes \boldsymbol{E}$ for the right quad $\hat{K}^\star$.

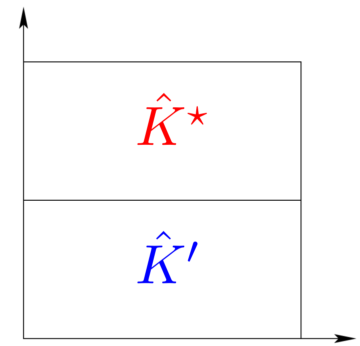


S Matrices: Tensor Product in 2D V

Horizontal subdivision:

$\boldsymbol{S}_{\hat{K}'\hat{K}} = \boldsymbol{E} \otimes \boldsymbol{S}_{\hat{J}'\hat{J}}$ for the bottom quad $\hat{K}'$,

$\boldsymbol{S}_{\hat{K}^\star\hat{K}} = \boldsymbol{E} \otimes \boldsymbol{S}_{\hat{J}^\star\hat{J}}$ for the top quad $\hat{K}^\star$.

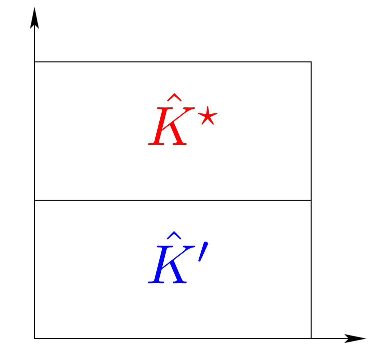


S Matrices: Tensor Product in 2D V

Horizontal subdivision:

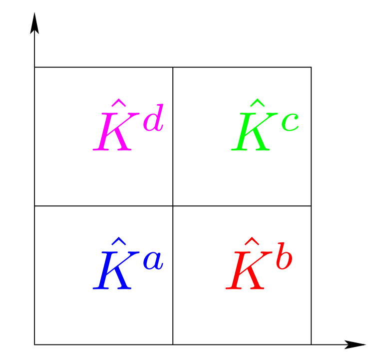
$$S_{\hat{K}'\hat{K}} = E \otimes S_{\hat{J}'\hat{J}} \quad \text{for the bottom quad } \hat{K}',$$

$$S_{\hat{K}^\star\hat{K}} = E \otimes S_{\hat{J}^\star\hat{J}} \quad \text{for the top quad } \hat{K}^\star.$$



Subdivision into four quads:

- subdivide $\hat{K}$ horizontally into two children
- subdivide upper and lower child vertically into $\hat{K}^d$ and $\hat{K}^c$ and $\hat{K}^a$ and $\hat{K}^b$ resp.



$$S_{\hat{K}^d\hat{K}} = \left(S_{\hat{J}'\hat{J}} \otimes E\right) \cdot \left(E \otimes S_{\hat{J}^\star\hat{J}}\right) \quad S_{\hat{K}^c\hat{K}} = \left(S_{\hat{J}^\star\hat{J}} \otimes E\right) \cdot \left(E \otimes S_{\hat{J}^\star\hat{J}}\right)$$

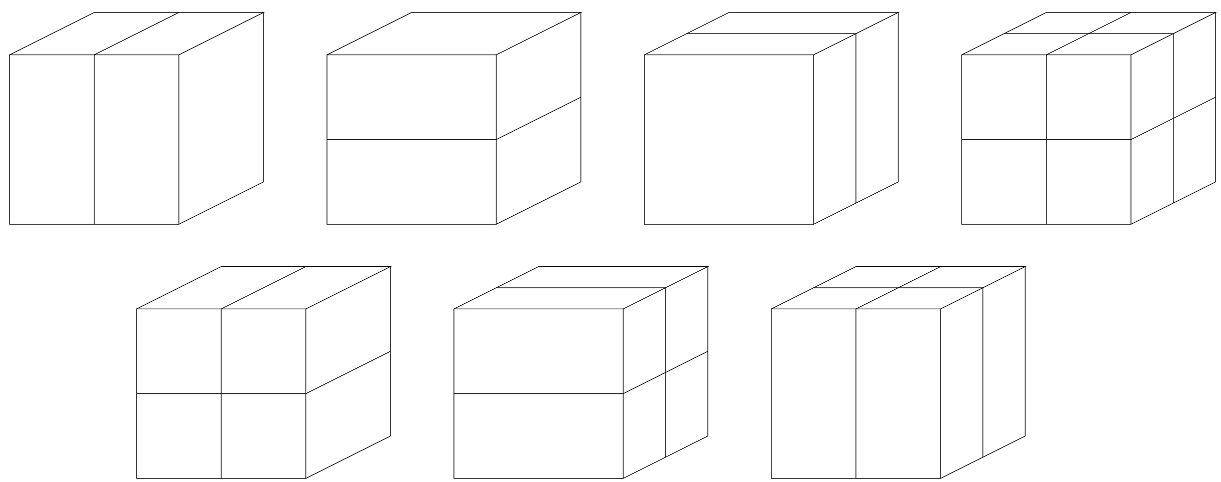
$$S_{\hat{K}^a\hat{K}} = \left(S_{\hat{J}'\hat{J}} \otimes E\right) \cdot \left(E \otimes S_{\hat{J}'\hat{J}}\right) \quad S_{\hat{K}^b\hat{K}} = \left(S_{\hat{J}^\star\hat{J}} \otimes E\right) \cdot \left(E \otimes S_{\hat{J}'\hat{J}}\right)$$

S Matrices: Tensor-Product in 3D

Same idea as in 2D, just of this form:

$$\boldsymbol{S}_{\hat{K}'\hat{K}} = \prod \left(\boldsymbol{A} \otimes \boldsymbol{B} \otimes \boldsymbol{C} \right)$$

in each of the factors, one of $\boldsymbol{A}$, $\boldsymbol{B}$ or $\boldsymbol{C}$ is an 1D S matrix.
Depending on the factors, 7 subdivisions are possible:



Concepts: allow arbitrary number and combination of these 7 subdivisions

in 3D.

Overview

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  - S and T matrices
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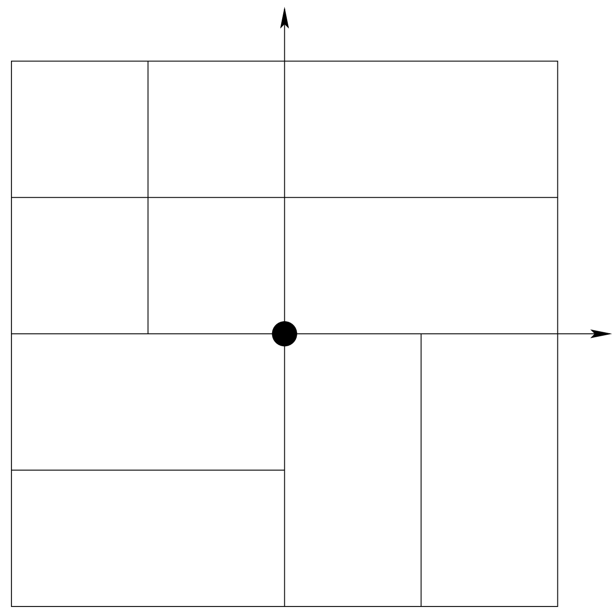
Anisotropic and Conforming

Main point: find the cells (either coarse or fine) which belong to the support of a certain basis function.



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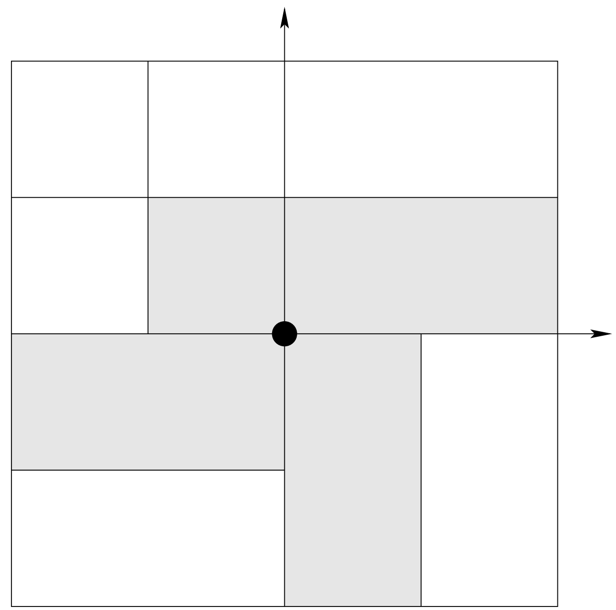

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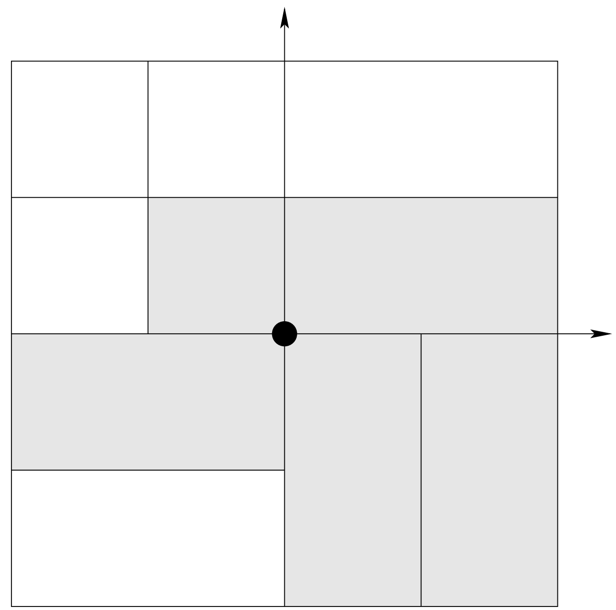

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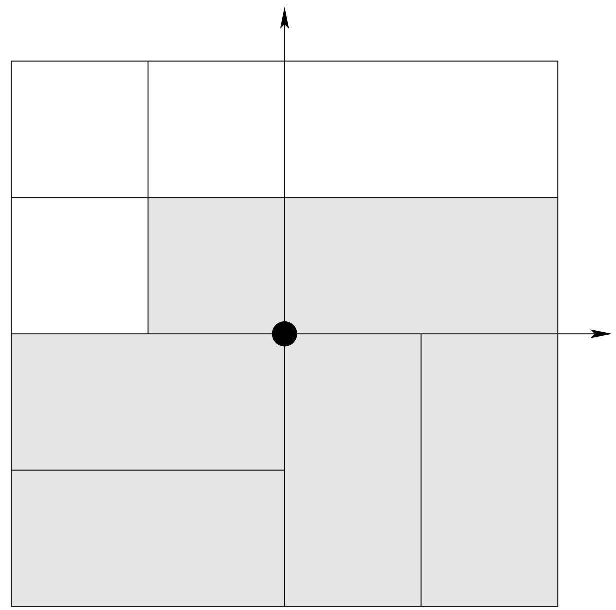

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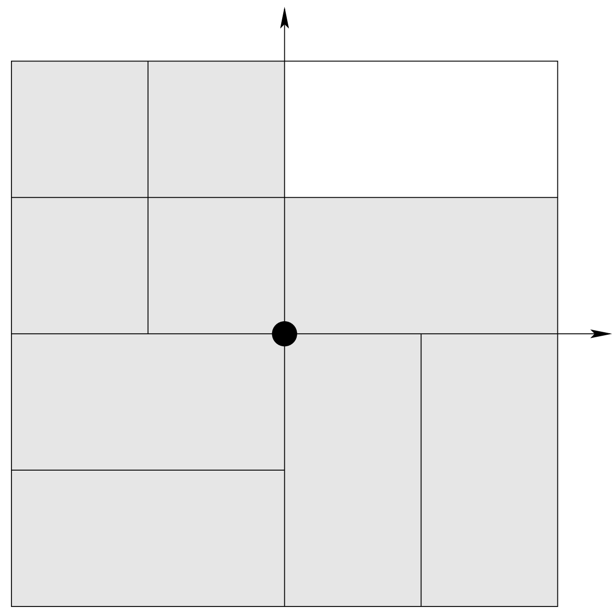

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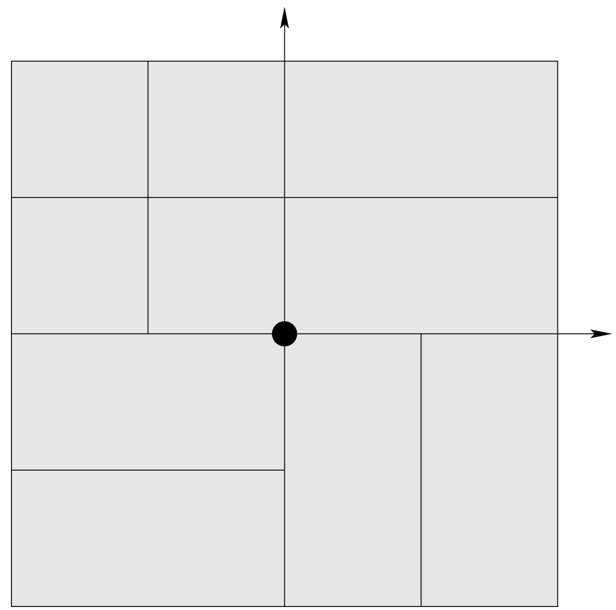
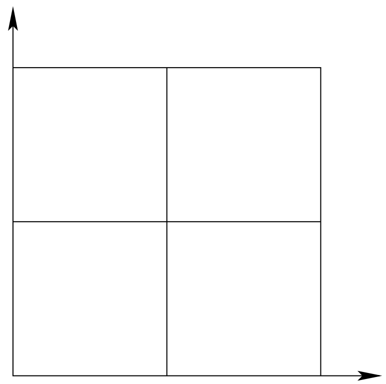
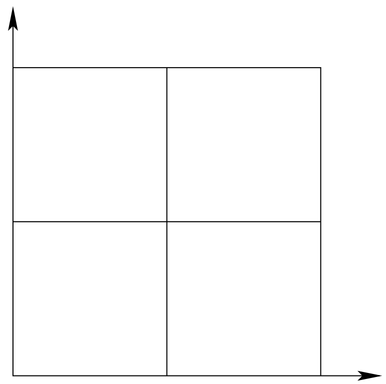


Anisotropic and Conforming II



- Can easily be treated since all edges are broken

Anisotropic and Conforming II



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- “Level of refinement” on each cell is enough to handle hanging nodes

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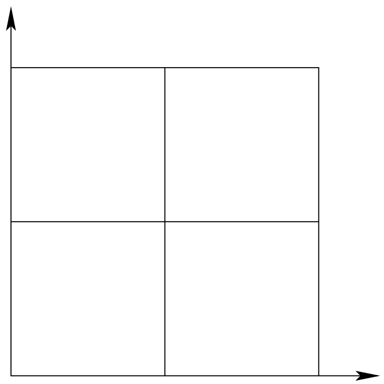

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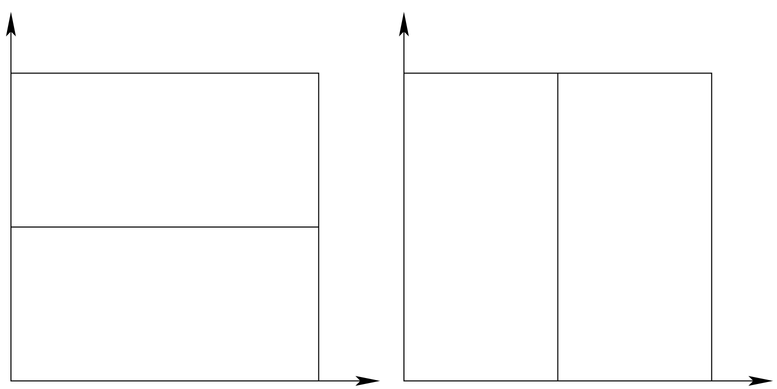

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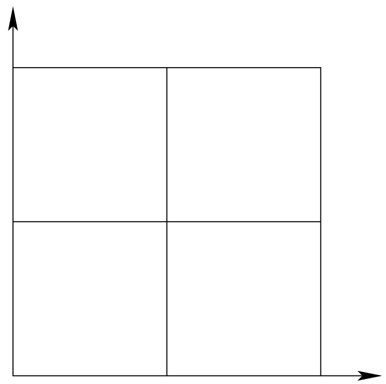

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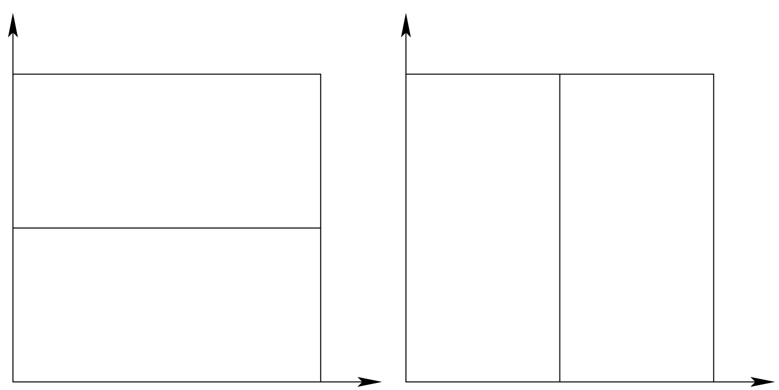

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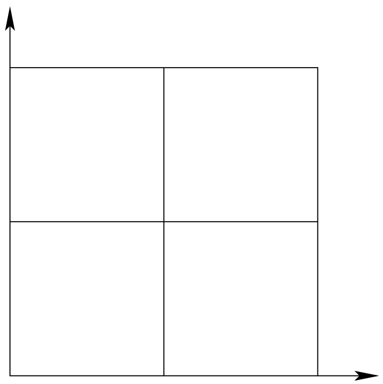
Anisotropic and Conforming II



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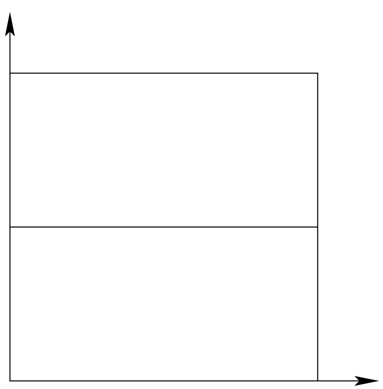

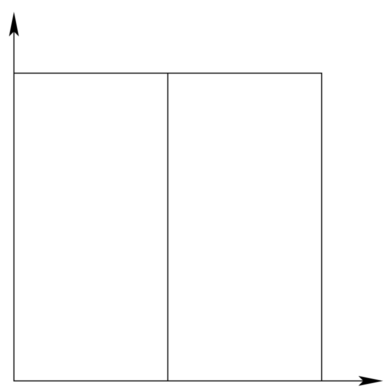

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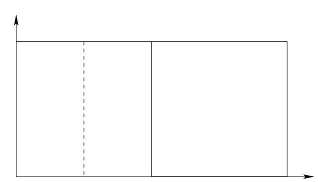
-  should be seen as conforming

Condition for Continuity

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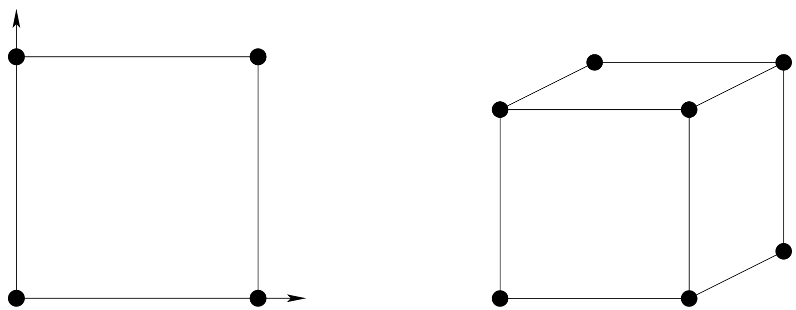

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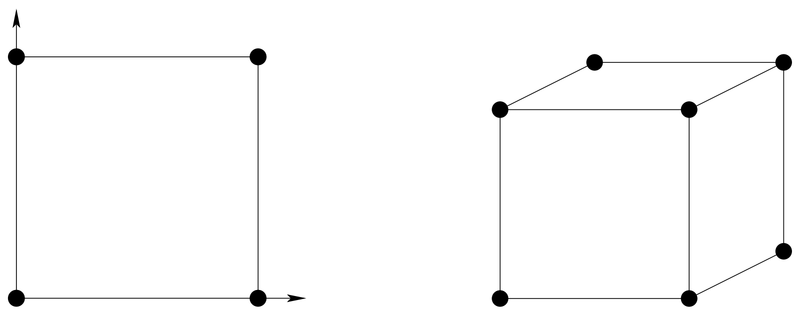


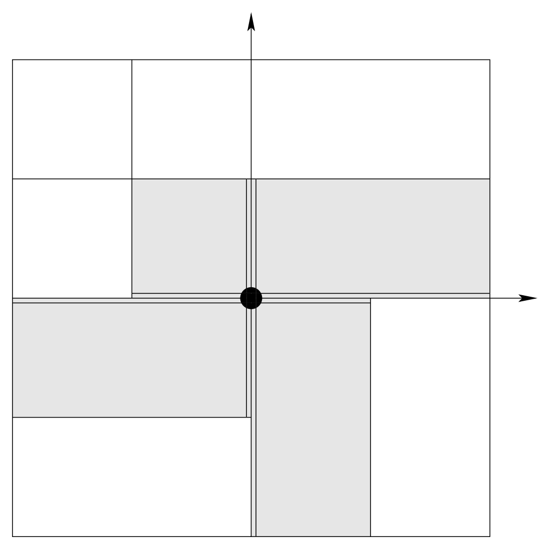
$\Rightarrow$ matching edges which coincide in a vertex is sufficient

- Unisolvent set for $\mathcal{Q}_p$ are
  - additional $p-1$ points on every edge
  - additional $(p-1)^2$ points on every face
  - additional $(p-1)^3$ points in the interior

$\Rightarrow$ matching faces which coincide in an edge is sufficient for continuous edge modes

Algorithm for Continuity (Vertex)

- In every cell of the finest mesh, register all edges and cells in their vertices.



Algorithm for Continuity (Vertex)

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- For every vertex, while something changed in the last loop:
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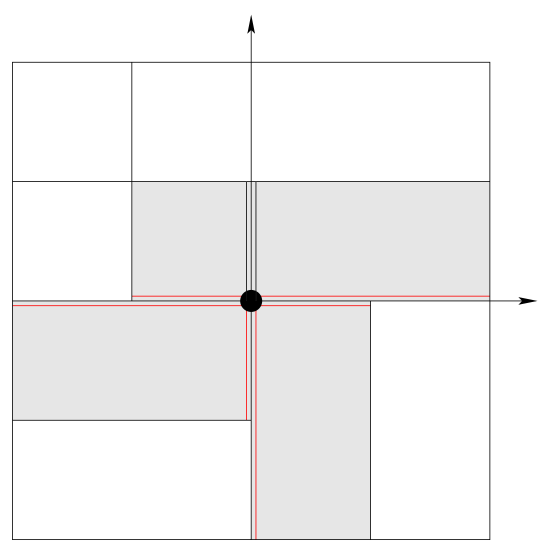

Algorithm for Continuity (Vertex)

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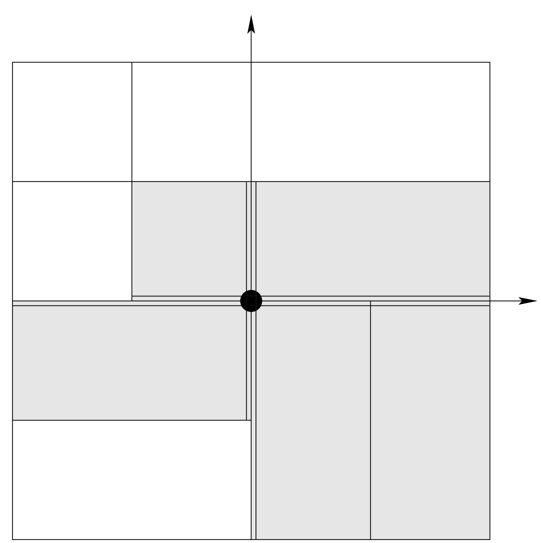
- For every vertex, while something changed in the last loop:
  - Check if some of the edges of the vertex have a relationship (ancestor / descendant).
  - If two edges are related, exchange the smaller cell in the list of the vertex by the cell matching the larger cell.
  - Delete the list of edges and rebuild it from the list of cells.



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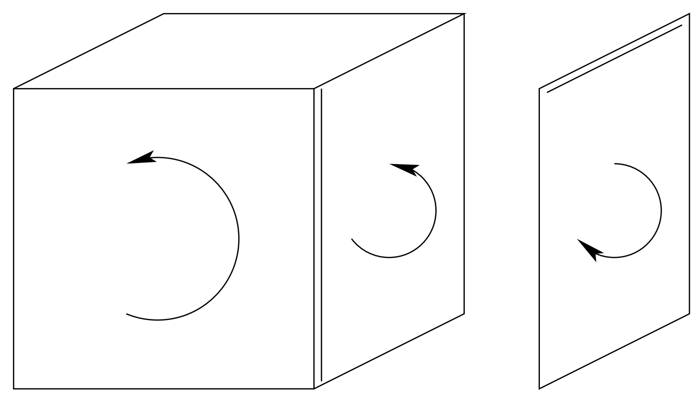
Anisotropic p

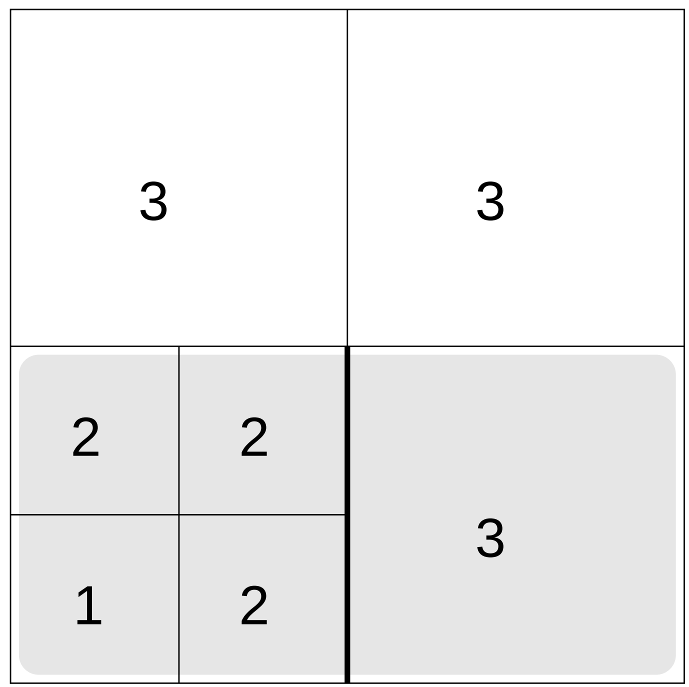
Why anisotropic p? Necessary for thin plates, shells, films.

- Every edge has p, every face has $\underline{p} = (p_0, p_1)$, every cell has $\underline{p} = (p_0, p_1, p_2)$. They can differ in a cell!
- Every face exists only once. It has a distinguished edge and an orientation. The orientation of the face in the cell and the position of the marked edge are stored: $\rho = 1$, $\tau = 3$.



p Enrichment on Edges

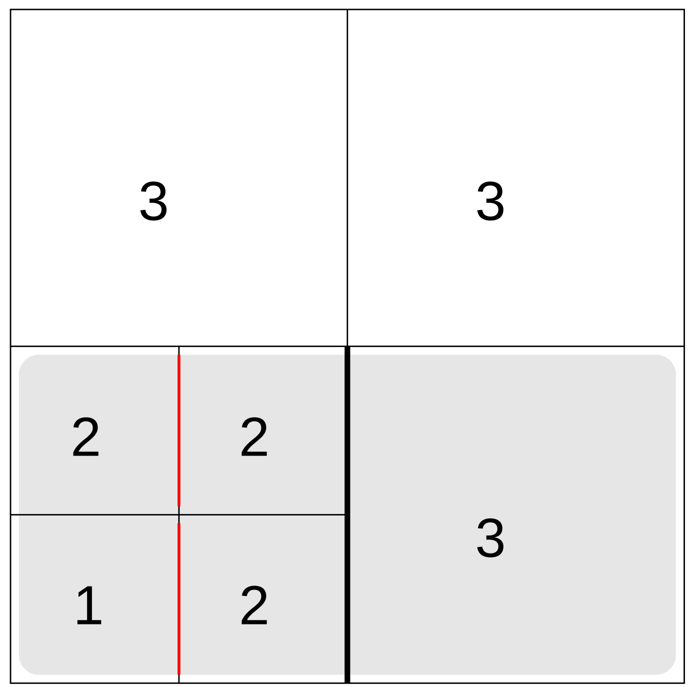
- $p^\star \geq 2$ for the basis functions Φ_i on the marked edge must be possible to achieve exponential convergence.



Analogly for edge and faces in 3D.

p Enrichment on Edges

- $p^\star \geq 2$ for the basis functions Φ_i on the marked edge must be possible to achieve exponential convergence.
- The basis functions on the red edges contribute to Φ_i $\Rightarrow p \geq p^\star$ must be possible and enforced.



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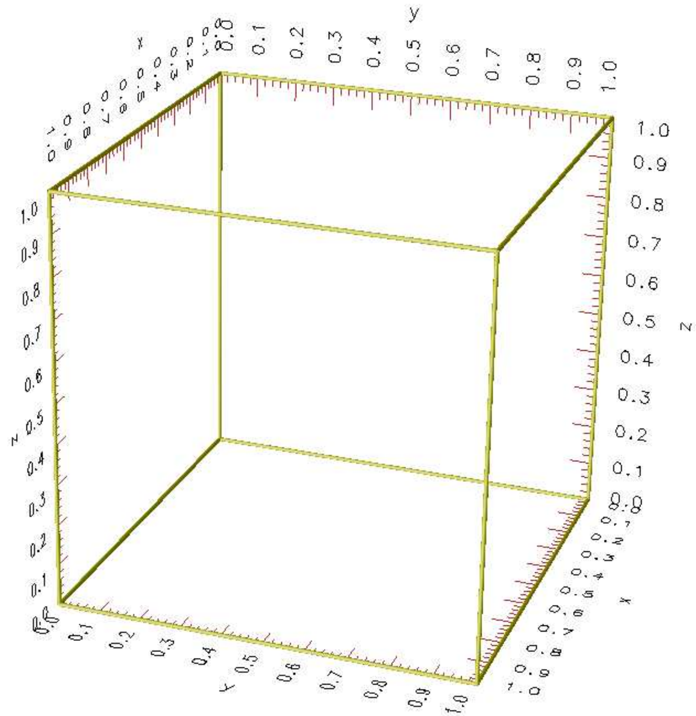
Some Basis Functions



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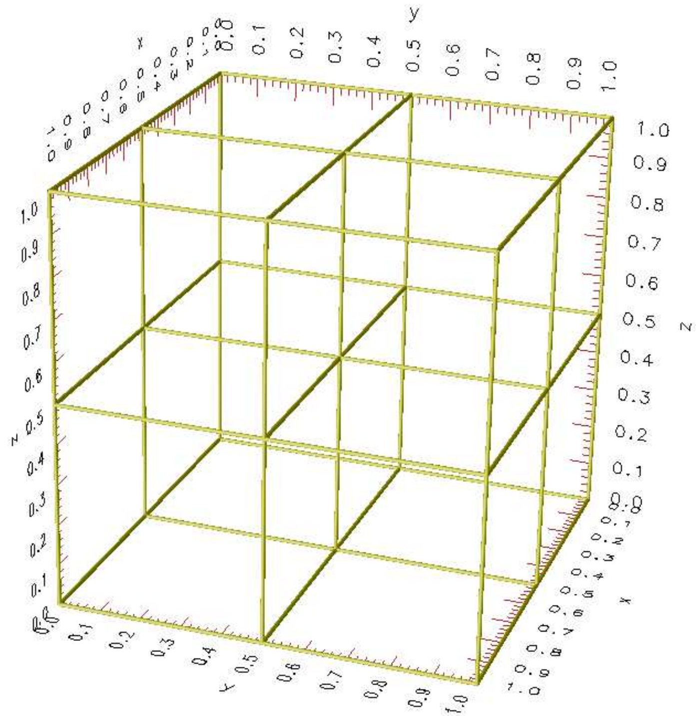

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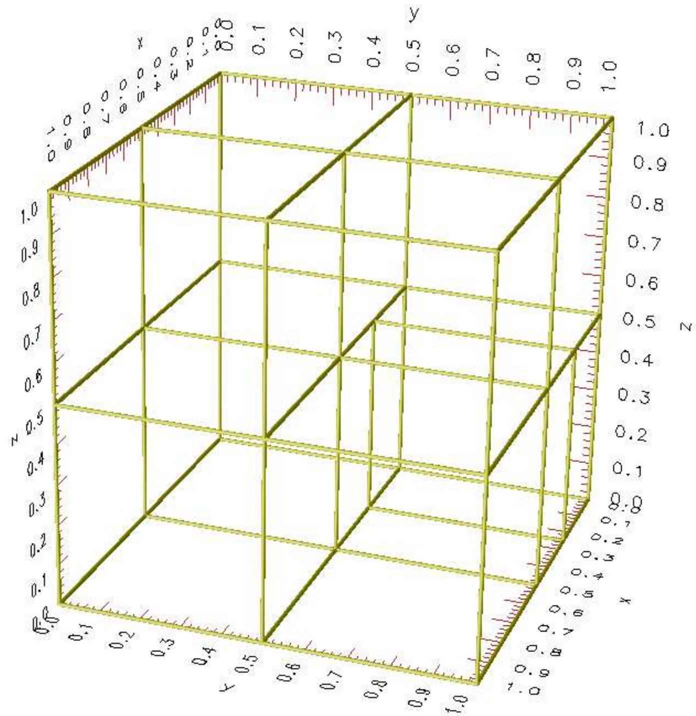

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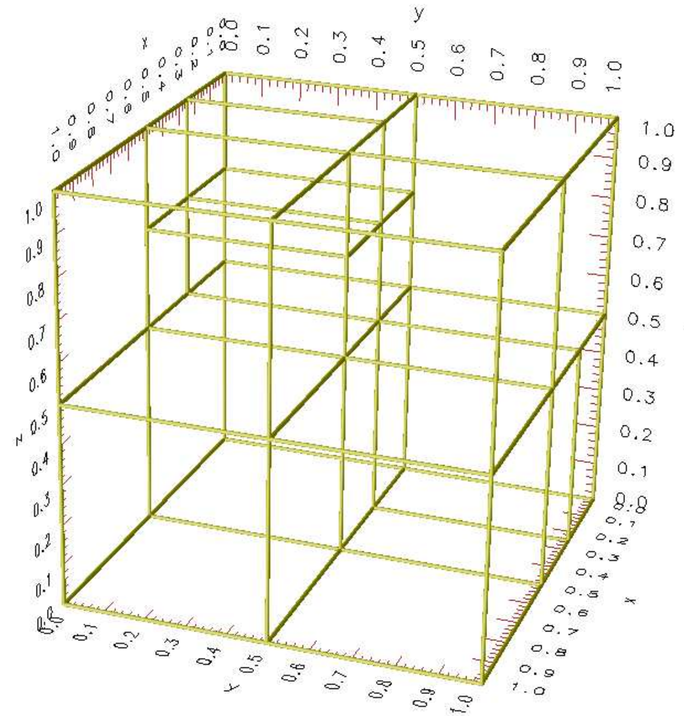

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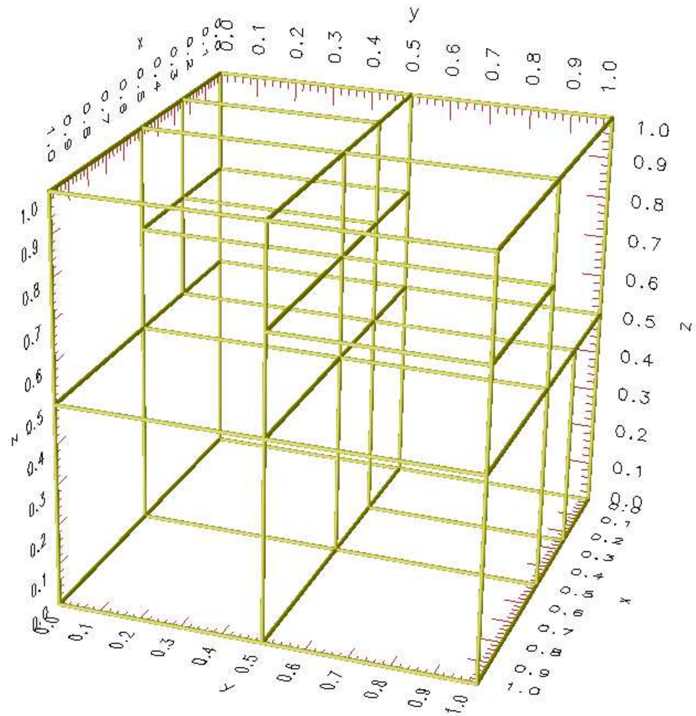
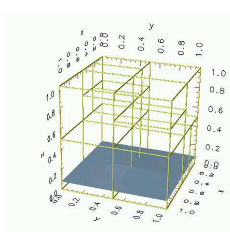
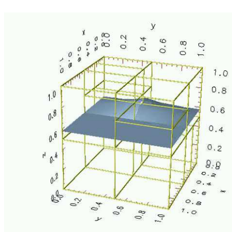
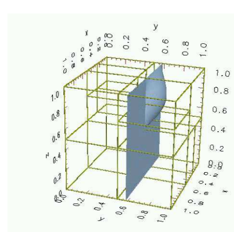


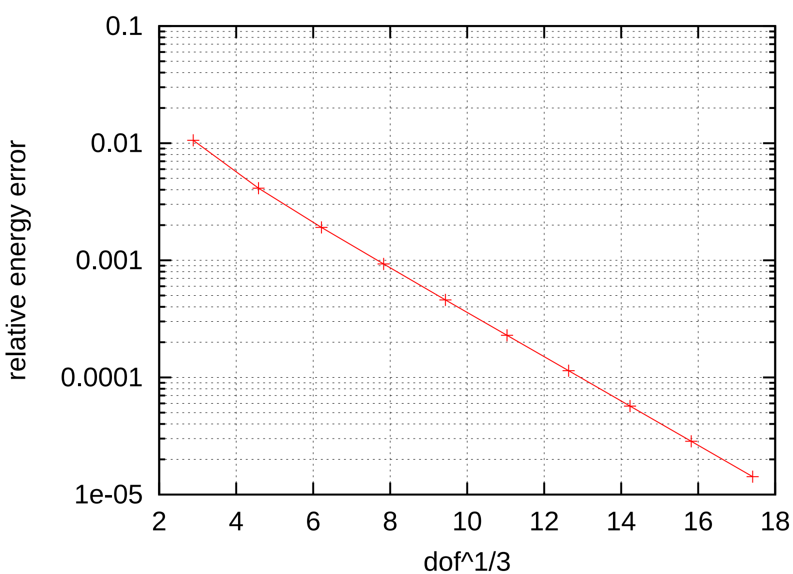


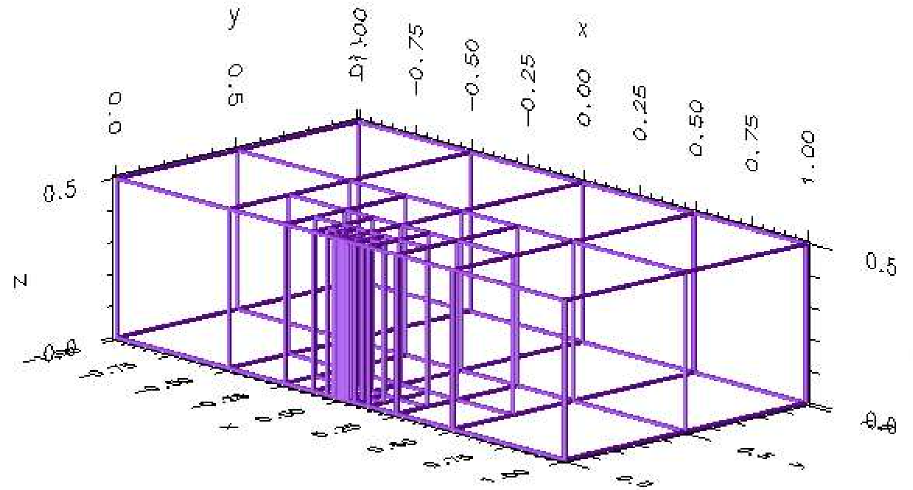




Exponential Convergence in Pseudo-3D

Edge type singularity.





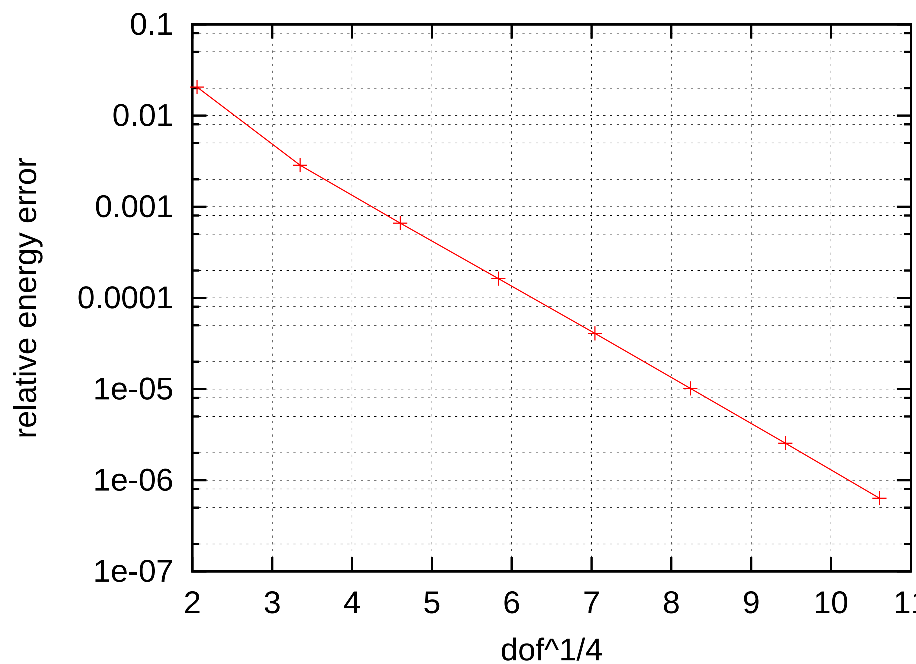
$$-\Delta u + u = f \text{ in } \Omega = (-1,1)\times(0,1)\times(0,{}^1\!/_2)$$

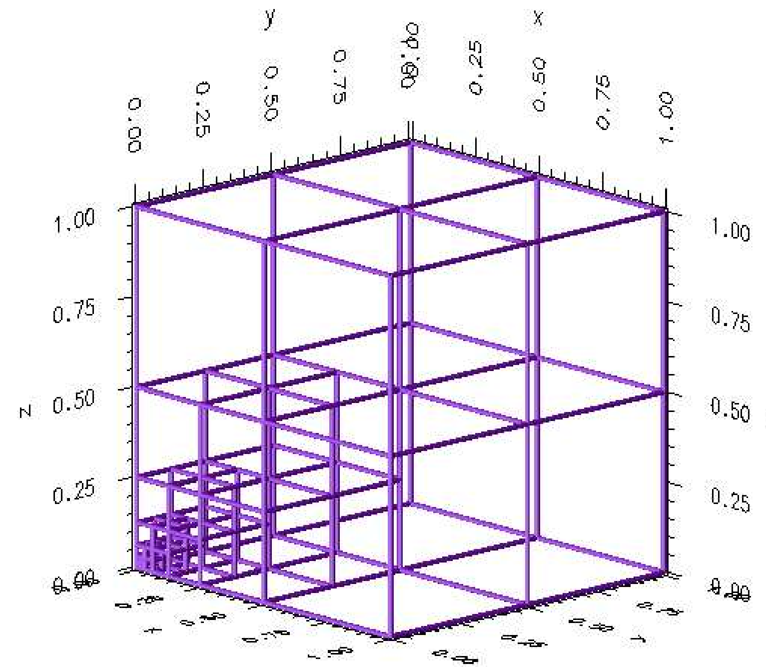
$$u(r,\phi,z) = \sqrt{r}\sin({}^{\phi}\!/_2)z(1-z) \qquad \text{in } \Omega$$

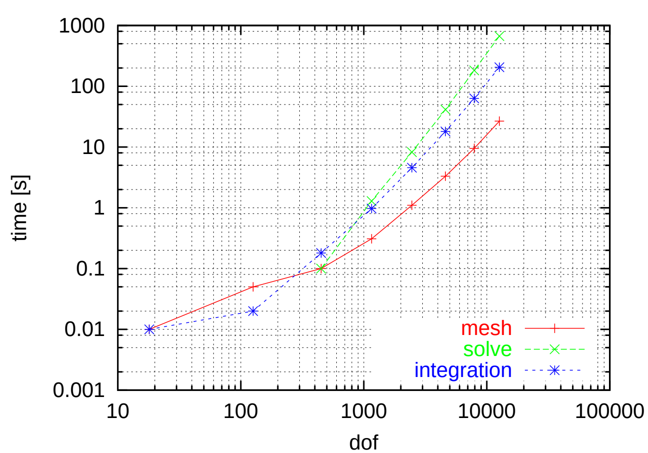
$$u = 0 \qquad \text{on } \{z=0\}\subset\partial\Omega$$

$$\text{and on } \{y=0\}\cap\{x\geq 0\}\subset\partial\Omega$$

Exponential Convergence in 3D







Vertex type singularity.

$$-\Delta u + u = f \text{ in } \Omega = (0,1)^3$$

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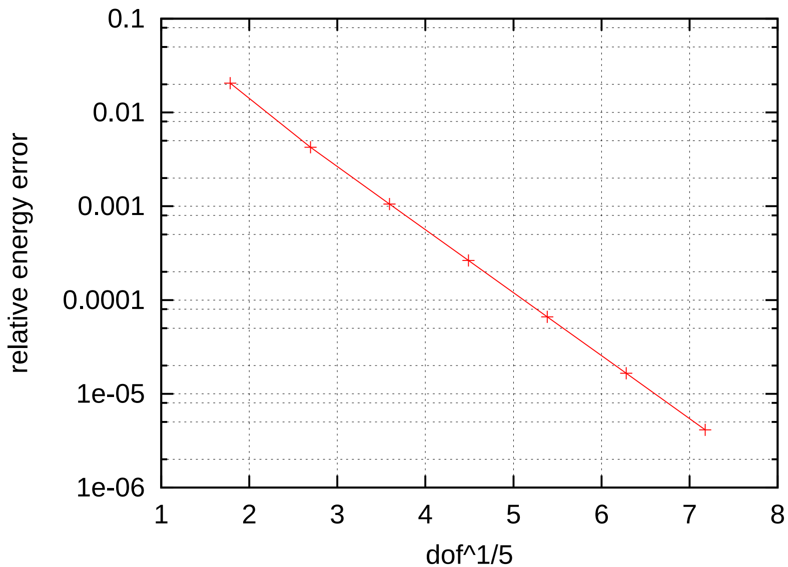
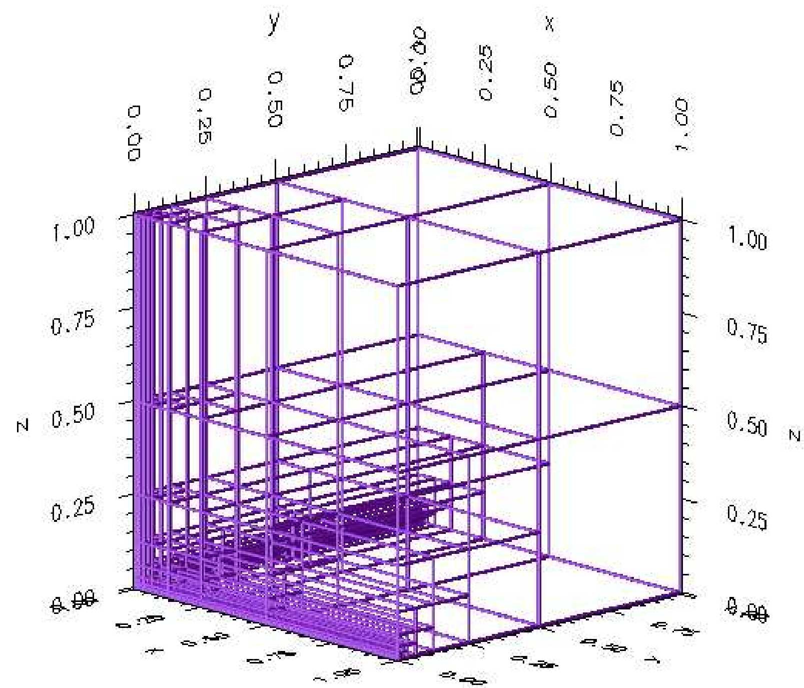
Exp. Conv. in 3D, Edge Mesh





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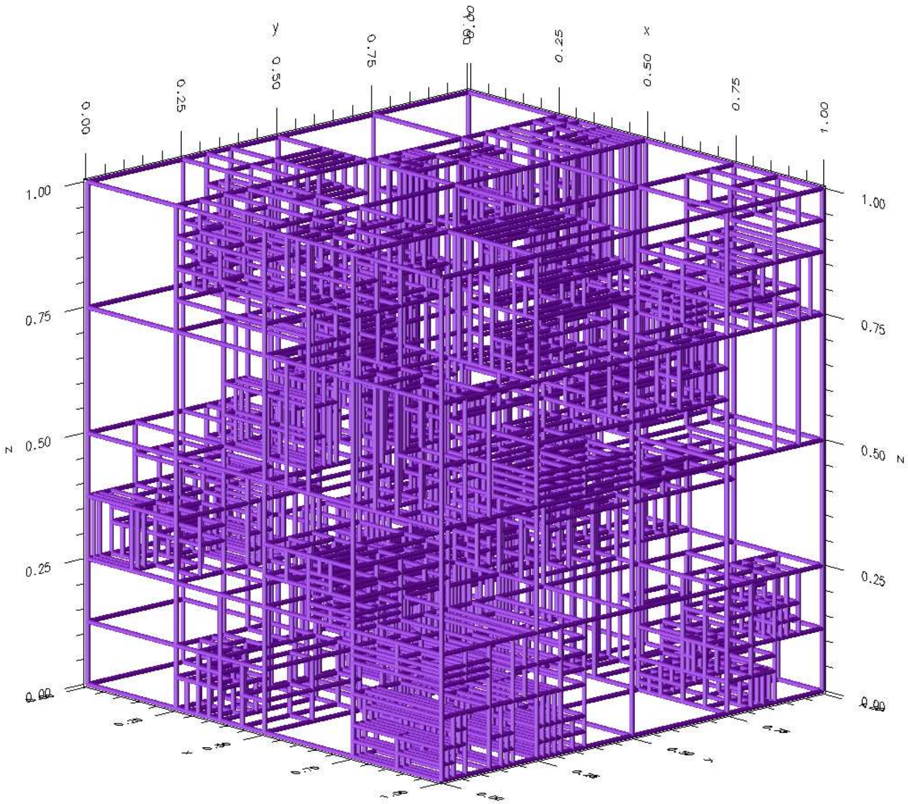
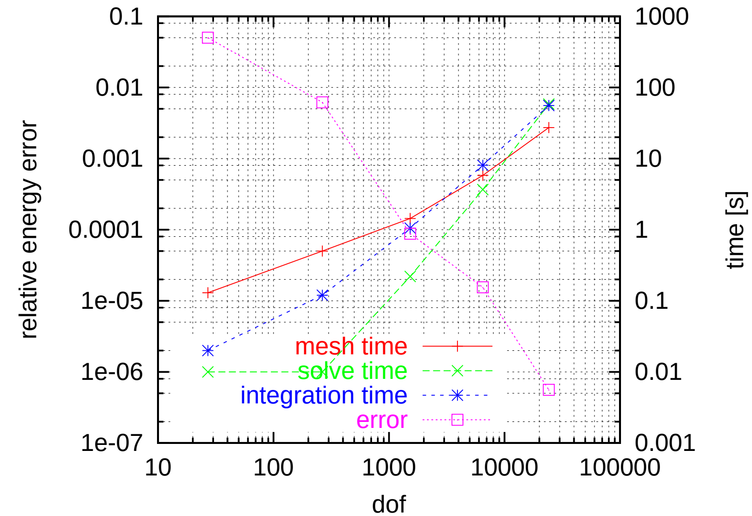
Test: Randomized Refinement





Refinements which are not allowed were dropped.

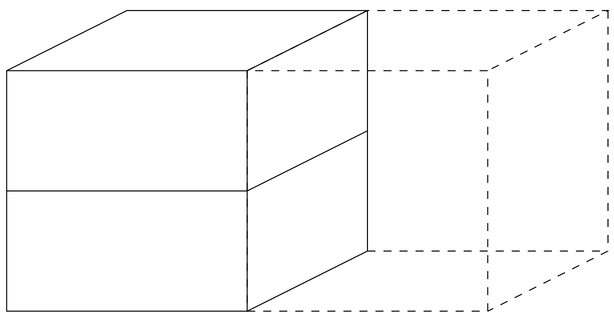
$$-\Delta u + u = f \text{ in } \Omega = (0,1)^3$$

$$u(x,y,z) = \sin(\pi x)\sin(\pi y)\sin(\pi z) \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

Perspectives

- Applications: Maxwell with weighted regularization (Costabel, Dauge)
- Anisotropic error estimation,
  anistropic regularity estimation
- Improved mesh handling



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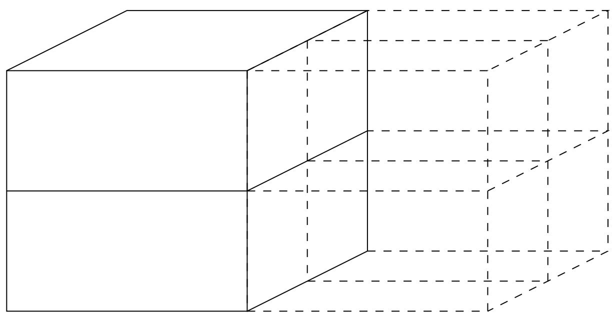

- Iterative multilevel domain decompositioning solvers:
  Toselli (Zürich), Schöberl (Linz)

Hanging Nodes in Isotropic Meshes

- Traverse all cells on locally finest level: mark every vertex / edge / face being used.
- On next (hierarchical) traversal of the mesh:
  - Add dofs which are marked to be on the current level to the list L of local dofs. Mark dof as registered.
  - If cell is on finest level $L \rightarrow$ T matrix
  - Otherwise $S \cdot L$ is added to L of child (next deeper level)

Mortar

- Give up $\mathcal{C}^0$, introduce Lagrange multiplier (the mortar)
- $-\Delta u = f$ in Ω with hom. Dirichlet bc. using mortar method leads to $\begin{pmatrix} \boldsymbol{A} & \boldsymbol{\Lambda} \\ \boldsymbol{\Lambda}^\top & 0 \end{pmatrix} \cdot \begin{pmatrix} \underline{u} \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}$, ie.

$$\begin{aligned} \text{SPD PDE} \;&\not\Rightarrow\; \text{SPD matrix} \\ &\Rightarrow\; \text{conjugate gradients not applicable} \\ &\Rightarrow\; \text{no standard domain decompositioning solvers} \\ &\Rightarrow\; \text{inf-sup condition needed} \end{aligned}$$

- The inf-sup cond. is OK in 2D, 3D for shape regular meshes.
  Not OK for hp FEM, existing proofs only for uniform meshes.
- Analogly for Discontinous Galerkin in 3D: Stability of hp DG on geometric meshes is not clear. First results by Schwab, Toselli, Schötzau for Stokes (not Mortar).